Frobenius morphisms and derived categories on two dimensional toric Deligne–Mumford stacks

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Abstract

For a toric Deligne–Mumford (DM) stack $\mathcal{X}$, we can consider a certain generalization of the Frobenius endomorphism. For such an endomorphism $F\colon \mathcal{X} \to \mathcal{X}$ on a 2-dimensional toric DM stack $\mathcal{X}$, we show that the push-forward $F_*\mathcal{O}_{\mathcal{X}}$ of the structure sheaf generates the bounded derived category of coherent sheaves on $\mathcal{X}$.

We also choose a full strong exceptional collection from the set of direct summands of $F_*\mathcal{O}_{\mathcal{X}}$ in several examples of two dimensional toric DM orbifolds $\mathcal{X}$.

1 Introduction

There are many smooth projective varieties X (or more generally algebraic stacks) whose derived categories $D^b(X)$ have elements generating $D^b(X)$ (see Definition 4.1). The direct sums of full exceptional collections give examples of such. Furthermore for a given vector bundle $\mathcal{E}$, if $\mathcal{E}$ generates $D^b(X)$ and it satisfies that $\mathrm{Hom}^i_{D^b(X)}(\mathcal{E},\mathcal{E}) = 0$ for $i \neq 0$, the derived category $D^b(X)$ is equivalent to the derived category of the module category of its endomorphism algebra $\mathrm{End}_{D^b(X)}(\mathcal{E})$ (cf. [TU, Lemma 3.3]), and then we can apply the representation theory of finite dimensional algebras to study $D^b(X)$. It is important to find a "good" generator of a given triangulated category.

In [Bo], Bondal announces that there exists a derived equivalence between abelian categories of coherent sheaves on a toric variety X and constructible sheaves on a real torus with a stratification associated to X. He uses the assertion (without proof) that the Frobenius push-forward $F_*\mathcal{O}_X$ of the structure sheaf generates the bounded derived category $D^b(X)$. The purpose of this note is to give a rigorous proof of it for the 2-dimensional stacky case:

Main Theorem (=Theorem 7.1)**.** *Let $\mathcal{X}$ be a two dimensional toric Deligne–Mumford (DM) stack. Then the vector bundle $F_*\mathcal{O}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$ for a Frobenius morphism F on $\mathcal{X}$ with a sufficiently divisible degree.*

Generators we find in Theorem 7.1 get along well with the birational geometry in some sense (cf. Lemma 4.4). Actually we make full use of the birational geometry to obtain Theorem 7.1. The proof of Theorem 7.1 is divided into 4 steps:

(i) First we reduce the proof to the case $\mathcal{X}$ is a toric DM orbifold. This step works for arbitrary dimensional case.

(ii) We introduce the notion of the *associated weighted blow up* on a toric DM orbifold $\mathcal{X}$ with a weighted blow up on a toric variety X which is the coarse moduli space of $\mathcal{X}$. In the two dimensional case we take a toric resolution of X, and consider the associated birational morphism on $\mathcal{X}$. We use Lemma 4.4 for this morphism to reduce the proof to the case X is smooth.

(iii) Use the strong factorization theorem to connect X and $\mathbb{P}^2$ by birational morphisms, and consider the associated birational morphisms. We use Lemma 4.4 and a technical Lemma 6.4 to reduce the proof to the case $\mathcal{X}$ is a root stack of $\mathbb{P}^2$.

(iv) Finally we show the statement for the case $\mathcal{X}$ is a root stack of $\mathbb{P}^2$.

The construction of this note is as follows. In §2 we recall the construction and some fundamental properties of toric DM stacks, following [BCS]. In §3 we introduce the notion of a *root stack* and the *rigidification* of toric DM stacks. In §4 we explain how to compute the direct summands of the Frobenius push-forward $F_*\mathcal{O}_{\mathcal{X}}$ for toric DM stacks $\mathcal{X}$, and put the step (i) into practice. In §5 we put the step (iv) into practice. In §6 we define the *associated birational morphisms* as in step (ii), and put the step (ii) into practice. In §7 we put the step (iii) into practice and complete the proof of Theorem 7.1. In §8 we use Theorem 7.1 to show existences of full strong exceptional collections in several examples.

We freely use terminology defined in §2.1 and §2.2 after these subsections. We always work over the complex number field $\mathbb{C}$, and note that (certain generalized) Frobenius morphisms can be defined on toric varieties over $\mathbb{C}$ (or actually any fields, see §4.2).

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Notation For any vector space $\mathbb{C}^s$ and any $m \in \mathbb{Z}_{>0}$, we denote by $\wedge m\colon \mathbb{C}^s \to \mathbb{C}^s$ a map defined by $\wedge m(z_1, \dots, z_s) = (z_1^m, \dots, z_s^m)$. We denote by $\mathbb{Z}_m$ the cyclic group $\mathbb{Z}/m\mathbb{Z}$. For any real vector $\boldsymbol{v} = {}^t(v_1, \dots, v_s) \in \mathbb{R}^s$, we put $\lfloor \boldsymbol{v} \rfloor = {}^t(\lfloor v_1 \rfloor, \dots, \lfloor v_s \rfloor) \in \mathbb{Z}^s$, where $\lfloor v_i \rfloor$ is the integer satisfying $\lfloor v_i \rfloor \leq v_i < \lfloor v_i \rfloor + 1$.

For a commutative ring R and $l, m \in \mathbb{Z}_{>0}$, we denote by $M(l, m, R)$ the set of $l \times m$ matrices in R. We often identify a matrix $A = (a_{i,j}) \in M(l, m, R)$ with a linear map

$$R^m \to R^l \quad {}^t(x_1, \dots, x_m) \mapsto A^t(x_1, \dots, x_m).$$

The symbol $\delta_{i,j}$ stands for the Kronecker delta. For $a_1, \dots, a_l \in R$, we denote by $\mathrm{diag}(a_1, \dots, a_l)$ the diagonal matrix $(a_i \delta_{i,j}) \in M(l, l, R)$.

For a group homomorphism $\pi\colon N \to N'$ between finitely generated abelian groups, we denote by $\pi_{\mathbb{R}}\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$ the associated $\mathbb{R}$-linear map between vector spaces $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $N'_{\mathbb{R}} = N' \otimes \mathbb{R}$. For a fan Δ in $N \otimes \mathbb{Q}$, $\Delta(1)$ denotes the set of 1-dimensional cones in Δ. X_Δ denote the toric variety associated with the fan Δ.

For a DM stack $\mathcal{X}$, we denote by $D^b(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$.

2 Toric DM stacks

Toric DM stacks are introduced in [BCS]. They are motivated by the Cox's construction of toric varieties ([Co]). In this section we recall some definitions and facts. See [FMN] and [Iw1] for other definitions.

2.1 Definitions

Let N be a finitely generated abelian group of rank n. We have an exact sequence of abelian groups

$$0 \to N_{\mathrm{tor}} \to N \to \bar{N} \to 0,$$

where N_{tor} is the subgroup of torsion elements in N. A *stacky fan* $\Sigma = (\Delta, \beta)$ in N consists of a simplicial fan Δ in $N \otimes \mathbb{Q}$ and a group homomorphism

$$\beta\colon \mathbb{Z}^s \to N$$

such that for the canonical basis $\boldsymbol{f}_1, \dots, \boldsymbol{f}_s$ of $\mathbb{Z}^s$, each $\beta(\boldsymbol{f}_i)$ generates the cone ρ_i in $N_{\mathbb{R}}$, where $\Delta(1) = \{\rho_1, \dots, \rho_s\}$. In this note we always assume that Δ is complete[1].

We define a toric DM stack associated to Σ as follows. We take the mapping cone $\mathrm{Cone}(\beta)$ of β in the derived category of $\mathbb{Z}$-modules and its derived dual $\mathrm{Cone}(\beta)^\star = \mathbf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cone}(\beta), \mathbb{Z})$. We put

$$\mathrm{DG}(\beta) := H^1(\mathrm{Cone}(\beta)^\star) \ \text{ and } G(= G_\Sigma) := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*).$$

[1]Completeness is not essential in many arguments below, but for simplicity, we assume it.

Remark 2.1. *Note that in the Cox's construction [Co], N is torsion free and every $\beta(\boldsymbol{f}_i)$ is primitive. In this case* $\mathrm{DG}(\beta)$ *is just the Chow group $A_{n-1}(X_\Delta)$ of the toric variety X_Δ.*

There exists an exact triangle

$$\mathrm{Cone}(\beta)^\star \to \mathbf{R}\,\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \to \mathbf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^s, \mathbb{Z})(\cong \mathbb{Z}^s),$$

which induces

$$\mathrm{cl}\colon \mathbb{Z}^s \to \mathrm{DG}(\beta). \tag{1}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to it, we obtain a map $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{cl}, \mathbb{C}^*)\colon G \to (\mathbb{C}^*)^s$. Hence by the natural action of $(\mathbb{C}^*)^s$ on $\mathbb{C}^s$, we have a G-action on $\mathbb{C}^s$.

Let $S := \mathbb{C}[z_1, \ldots, z_s]$ be the coordinate ring of $\mathbb{C}^s$. For each cone σ in Δ, we put $z_\sigma = \prod_{\beta_{\mathbb{R}}(\boldsymbol{f}_i)\notin\sigma} z_i$ and $U_\sigma := \mathbb{C}^s \setminus \{z_\sigma = 0\}$. We have a G-invariant subspace

$$U(= U_\Sigma) := \bigcup_{\sigma\in\Delta} U_\sigma.$$

The *toric Deligne–Mumford (DM) stack $\mathcal{X} = \mathcal{X}_\Sigma$ associated to the stacky fan* Σ is defined by the quotient stack

$$\mathcal{X} := [U/G].$$

This is actually a DM stack by [BCS, Proposition 3.2].

When N is torsion free, the stack $\mathcal{X}$ has the trivial generic stabilizer group, and we call $\mathcal{X}$ a *toric DM orbifold*. In this case, let us denote by $\boldsymbol{v}_i$ the primitive vectors in $\rho_i \in \Delta(1)$, and also denote by D_i (resp. $\mathcal{D}_i$) the toric divisors on $X = X_\Delta$ (resp. $\mathcal{X} = \mathcal{X}_\Sigma$) corresponding to the cone ρ_i (see their precise definitions in §2.2 and §2.3). Take the positive integers b_i satisfying $\beta(\boldsymbol{f}_i) = b_i\boldsymbol{v}_i$. We often denote the toric DM orbifold $\mathcal{X}$ by

$$\mathcal{X}\bigl(X, \textstyle\sum b_i D_i\bigr).$$

Remark 2.2. *If N is torsion free, then the map* $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{cl}, \mathbb{C}^*)$ *becomes an inclusion. Assume furthermore that every $\beta(\boldsymbol{f}_i)$ is primitive and the fan Δ is non-singular, i.e. every cone is generated by a subset of a basis of N. Then we can see that G acts on U freely, and actually the converse is also true. In this case, $\mathcal{X}$ is an algebraic space. Since the toric variety $X = X_\Delta$ is a coarse moduli space of $\mathcal{X}$ ([BCS, Proposition 3.7]), $\mathcal{X}$ coincides with X. Thus in this situation, the construction of toric DM stacks is same as the original construction of toric varieties given in [Co].*

Note that the orbifold $\mathcal{X}\bigl(X, \sum D_i\bigr)$ coincides with its coarse moduli space X if and only if X is smooth. In general the orbifold $\mathcal{X}\bigl(X, \sum D_i\bigr)$ is the canonical stack with the coarse moduli space X. We put $\mathcal{X}^{can} = \mathcal{X}\bigl(X, \sum D_i\bigr)$.

We denote by $\mathrm{Coh}_G U$ the category of G-equivariant coherent sheaves on U. Then we have an equivalence of categories

$$\mathrm{Coh}\,\mathcal{X} \cong \mathrm{Coh}_G U \quad \mathcal{F} \mapsto \mathcal{F}_U \tag{2}$$

by [Vi, (7.21)]. By this equivalence we identify $\mathrm{Coh}\,\mathcal{X}$ and $\mathrm{Coh}_G U$, and define

$$H^0(U, \mathcal{F}) := H^0(U, \mathcal{F}_U)$$

for any $\mathcal{F} \in \mathrm{Coh}\,\mathcal{X}$.

Define $G^\vee$ to be the group of characters on G, that is, $G^\vee := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$, and then note that the map cl in (1) can be regarded as the map from $\mathbb{Z}^s$ to $G^\vee$, since $G^\vee$ is naturally isomorphic to $\mathrm{DG}(\beta)$. The G-action on $\mathbb{C}^s = \mathrm{Spec}\,S$ induces an eigenspace decomposition

$$S = \bigoplus_{\chi \in G^\vee} S_\chi, \text{ where we put } S_\chi := \bigoplus_{\substack{\boldsymbol{k} \in \mathbb{Z}^s_{\geq 0} \\ \mathrm{cl}(\boldsymbol{k}) = \chi}} \mathbb{C} z^{\boldsymbol{k}} \tag{3}$$

and $z^{\boldsymbol{k}} := z_1^{k_1} \cdots z_s^{k_s}$ for $\boldsymbol{k} = {}^t(k_1, \cdots, k_s) \in \mathbb{Z}^s$. We call the $G^\vee$-graded ring S the *homogeneous coordinate ring* of $\mathcal{X}$. The point in $\mathcal{X}$ corresponding to the G-orbit of $(z_1, \ldots, z_s) \in U$ is denoted by $[z_1 : \cdots : z_s]$. We call it the *homogeneous coordinate* of $\mathcal{X}$.

We denote by $\mathrm{gr}S$ the category of $G^\vee$-graded finitely generated S-modules and by $\mathrm{tor}\,S$ the full subcategory of $\mathrm{gr}S$ consisting of S-modules annihilated by some powers of the ideal $(z_\sigma \mid \sigma \in \Delta)$. We associate a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ with a $G^\vee$-graded S-module

$$M := \bigoplus_{\chi \in G^\vee} H^0(U, \mathcal{F})_\chi,$$

where $H^0(U, \mathcal{F})_\chi$ is the set of sections having an eigenvalue $\chi \in G^\vee$ for the action of G. The module M is finitely generated by [BH1, Lemma 4.7] and a unique $G^\vee$-graded S-module, up to $\mathrm{tor}\,S$, satisfying $\tilde{M}|_U \cong \mathcal{F}_U$ in $\mathrm{Coh}_G U$. Hence we have an equivalence of categories

$$\mathrm{Coh}\,\mathcal{X} \cong \mathrm{gr}S/\,\mathrm{tor}\,S \quad \mathcal{F} \mapsto M = \bigoplus_{\chi \in G^\vee} H^0(U, \mathcal{F})_\chi. \tag{4}$$

Here $\mathrm{gr}S/\,\mathrm{tor}\,S$ is the quotient category (cf. [HMS, Appendix A.2]).

2.2 Picard groups and toric divisors of toric DM stacks

For $\chi \in G^\vee = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$, we define a G-action on the trivial bundle $U \times \mathbb{C}$ by

$$G \times U \times \mathbb{C} \to U \times \mathbb{C} \quad (g, z, t) \mapsto (gz, \chi(g)t). \tag{5}$$

We obtain a G-equivariant trivial line bundle on U, which defines a line bundle $\mathcal{L}_\chi$ on $\mathcal{X}$ via the equivalence (2). This gives an isomorphism $G^\vee \cong \operatorname{Pic}\mathcal{X}$ as [BHu, Proposition 3.3]. Note that Borisov and Hua show this isomorphism under the assumption $N_{tor} = 0$, but a similar proof works in the case $N_{tor} \neq 0$. Henceforth we often identify $\operatorname{Pic}\mathcal{X}$, $G^\vee$ and $\mathrm{DG}(\beta)$, and their elements corresponding to each other are denoted as

$$\mathrm{DG}(\beta) \cong G^\vee \cong \operatorname{Pic}\mathcal{X} \qquad \bar{\boldsymbol{w}} \mapsto \chi_{\overline{\boldsymbol{w}}} = \chi \mapsto \mathcal{L}_\chi. \tag{6}$$

We describe $\operatorname{Pic}\mathcal{X}$ more explicitly. Take a projective resolution of N;

$$0 \to \mathbb{Z}^r \xrightarrow{A} \mathbb{Z}^{n+r} \to N \to 0$$

for a matrix $A = (a_{i,j}) \in M(n+r, r, \mathbb{Z})$. Then the map $\beta\colon \mathbb{Z}^s \to N$ is determined by some matrix $B = (b_{i,j}) \in M(n+r, s, \mathbb{Z})$, and the mapping cone of β is described as

$$\mathrm{Cone}(\beta) = \{\cdots \to 0 \to \mathbb{Z}^s \oplus \mathbb{Z}^r \xrightarrow{(B|A)} \mathbb{Z}^{n+r} \to 0 \to \cdots\},$$

where the term $\mathbb{Z}^{n+r}$ fits into the degree 0 part. Let us denote by $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s$ and $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r$ the canonical bases of $\mathbb{Z}^s$ and $\mathbb{Z}^r$ respectively. Since

$$\mathrm{DG}(\beta) = H^1(\mathrm{Cone}(\beta)^\star) = \operatorname{coker}{}^t(B|A), \tag{7}$$

we have

$$\operatorname{Pic}\mathcal{X} \cong \mathrm{DG}(\beta) = \frac{\mathbb{Z}\boldsymbol{f}_1^* \oplus \cdots \oplus \mathbb{Z}\boldsymbol{f}_s^* \oplus \mathbb{Z}\boldsymbol{g}_1^* \oplus \cdots \oplus \mathbb{Z}\boldsymbol{g}_r^*}{\left\langle \sum_{j=1}^s b_{i,j}\boldsymbol{f}_j^* + \sum_{j=1}^r a_{i,j}\boldsymbol{g}_j^* \mid i = 1, \ldots, n+r \right\rangle}. \tag{8}$$

We denote by $\bar{\boldsymbol{f}}_i^*$ and $\bar{\boldsymbol{g}}_j^*$ the classes of dual bases $\boldsymbol{f}_i^*$ and $\boldsymbol{g}_j^*$ in $\mathrm{DG}(\beta)$ respectively.

By (7), the group $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$ is described as

$$G = \left\{ \boldsymbol{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_{s+r} \end{pmatrix} \in (\mathbb{C}^*)^{s+r} \;\middle|\; \prod_{j=1}^s t_j^{b_{i,j}} \times \prod_{j=1}^r t_{s+j}^{a_{i,j}} = 1 \text{ for } i = 1, \ldots, n+r \right\}. \tag{9}$$

The G-action on $U \subset \mathbb{C}^s$ is defined by

$$\boldsymbol{t} \cdot (z_1, \ldots, z_s) = (t_1 z_1, \ldots, t_s z_s) \tag{10}$$

for $\boldsymbol{t} \in G$. Hence we can see that the group

$$H(= H_\Sigma) := \{\boldsymbol{t} \in G \mid t_1 = \cdots = t_s = 1\} \cong \left\{ \begin{pmatrix} t_{s+1} \\ \vdots \\ t_{s+r} \end{pmatrix} \in (\mathbb{C}^*)^r \;\middle|\; \prod_{j=1}^r t_{s+j}^{a_{i,j}} = 1 \text{ for } i = 1, \ldots, n+r \right\}$$

acts on U trivially, and it is the generic stabilizer group of $\mathcal{X} = [U/G]$. Note that H is a finite group, since $\operatorname{rk} A = r$.

For each $i = 1, \dots, s$, we have a Cartier divisor $\mathcal{D}_i (= \mathcal{D}_i^{\mathcal{X}}) := [\{z_i = 0\}/G]$ (in the notation of §2.3, $\mathcal{D}_i = [z_i = 0]$) on $\mathcal{X} = [U/G]$ corresponding to the ray ρ_i, which satisfies that $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i) \cong \mathcal{L}_{\chi_{\bar{\boldsymbol{f}}_i^*}}$ in the notation in (6). We call $\mathcal{D}_i$ the *toric divisors corresponding to the cone ρ_i*. When $\mathcal{X}$ is a variety, we often denote $\mathcal{D}_i$ by D_i. Put

$$\boldsymbol{D} (= \boldsymbol{D}^{\mathcal{X}}) := \begin{pmatrix} \mathcal{D}_1 \\ \vdots \\ \mathcal{D}_s \end{pmatrix} \text{ and } \boldsymbol{k}\boldsymbol{D} = \sum_{i=1}^{s} k_i \mathcal{D}_i$$

for $\boldsymbol{k} = \sum_{i=1}^{s} k_i \boldsymbol{f}_i^* \in \mathbb{Z}^s$. We consider a $G^\vee$-graded S-module $Sz^{-\boldsymbol{k}}$ (recall that $z^{-\boldsymbol{k}} = z_1^{-k_1} \cdots z_s^{-k_s}$ as in (3)) corresponding to $\mathcal{O}_{\mathcal{X}}(\boldsymbol{k}\boldsymbol{D})$ via the isomorphism in (4), where the $G^\vee$-grading of $Sz^{-\boldsymbol{k}}$ is given by $\mathrm{cl}\colon \mathbb{Z}^s \to G^\vee$ as in (3).

For $\boldsymbol{w} \in \mathbb{Z}^s \oplus \mathbb{Z}^r$ and a $G^\vee$-graded S-module M, we define a $G^\vee$-graded S-module

$$M(\boldsymbol{w}) = \bigoplus_{\chi \in G^\vee} M(\boldsymbol{w})_\chi$$

by $M(\boldsymbol{w})_\chi := M_{\chi + \chi_{\bar{\boldsymbol{w}}}}$, where $\bar{\boldsymbol{w}}$ is the image of $\boldsymbol{w}$ in $\mathrm{DG}(\beta)$, and $\chi_{\bar{\boldsymbol{w}}} \in G^\vee$ is defined in (6). Then we have an isomorphism of $G^\vee$-graded S-modules $Sz^{-\boldsymbol{k}} \cong S(\boldsymbol{k})$.

For $\boldsymbol{l} \in \mathbb{Z}^r$ denote by

$$\mathcal{O}_{\mathcal{X}}(\boldsymbol{k}\boldsymbol{D})_{\boldsymbol{l}}$$

the line bundle which corresponds to the graded S-module $S(\boldsymbol{k} + \boldsymbol{l})$ by (4). Here we regard $\boldsymbol{k} + \boldsymbol{l}$ as an element of $\mathbb{Z}^s \oplus \mathbb{Z}^r$. Note that $\mathcal{O}_{\mathcal{X}}(\boldsymbol{k}\boldsymbol{D})_{\boldsymbol{l}}$ is the line bundle $\mathcal{L}_{\chi_{\overline{\boldsymbol{k}+\boldsymbol{l}}}}$ in (6). When $\boldsymbol{k} = \boldsymbol{0}$, we put $\mathcal{O}_{\mathcal{X},\boldsymbol{l}} = \mathcal{O}_{\mathcal{X}}(\boldsymbol{0}\boldsymbol{D})_{\boldsymbol{l}}$.

Henceforth we freely use terminology defined in §2.1 and 2.2. We often use the superscript $'$ for objects associated with a toric DM stack $\mathcal{X}'$. For instance, $\Sigma' = (\Delta', \beta')$ stands for the stacky fan in finitely generated abelian group N' defining $\mathcal{X}'$.

2.3 Closed substacks of toric DM stacks

Let $\Sigma = (\Delta, \beta)$ be a stacky fan and $\mathcal{X} = \mathcal{X}_\Sigma$ the associated toric DM stack. For any non-zero cone $\tau \in \Delta$, we consider the abelian group

$$N(\tau) := \frac{N}{\langle \beta(\boldsymbol{f}_i) \mid \rho_i \subset \tau \rangle}$$

and denote by $\pi_\tau \colon N \to N(\tau)$ the quotient map. We consider the fan

$$\Delta_\tau := \left\{\pi_{\tau,\mathbb{R}}(\sigma) \mid \sigma + \tau \in \Delta, \sigma \in \Delta\right\} = \left\{\pi_{\tau,\mathbb{R}}(\sigma) \;\middle|\; \tau \subset \sigma \in \Delta\right\}$$

in $N(\tau)_\mathbb{R}$. Then the map $\pi_{\tau,\mathbb{R}}$ gives a one to one correspondence between the set

$$\left\{\rho_i \;\middle|\; \rho_i + \tau \in \Delta, \rho_i \cap \tau = \mathbf{0}\right\}$$

and the set

$$\Delta_\tau(1) = \left\{\pi_{\tau,\mathbb{R}}(\rho_i) \;\middle|\; \rho_i + \tau \in \Delta, \rho_i \cap \tau = \mathbf{0}\right\}.$$

By this identification, we regard the set $\Delta_\tau(1)$ as a subset of $\Delta(1)$, and hence we have $\mathbb{Z}^{\Delta_\tau(1)} \subset \mathbb{Z}^{\Delta(1)} = \mathbb{Z}^s$ under the identifications $\mathbb{Z}^{\Delta_\tau(1)} \cong \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{\Delta_\tau(1)}, \mathbb{Z})$ and $\mathbb{Z}^{\Delta(1)} \cong \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z})$. Then we define a stacky fan $\Sigma_\tau = (\Delta_\tau, \beta_\tau)$ in $N(\tau)$, where

$$\beta_\tau := \pi_\tau \circ (\beta|_{\mathbb{Z}^{\Delta_\tau(1)}}).$$

The toric DM stack $\mathcal{X}_{\Sigma_\tau}$ associated to the stacky fan Σ_τ defines a closed substack of $\mathcal{X}$ as follows.

Reorder the set $\Delta(1)$ so that $\Delta_\tau(1) = \{\pi_{\tau,\mathbb{R}}(\rho_1), \dots, \pi_{\tau,\mathbb{R}}(\rho_{s_\tau})\}$ and $\tau = \rho_{s_\tau+1} + \cdots + \rho_l$ for some s_τ, l with $l > s_\tau \geq 0$. For the homogeneous coordinate $[z_1 : \cdots : z_s]$ of $\mathcal{X}$, we define a subset V_τ of U and a subgroup G_τ of G by

$$V_\tau := \{{}^t(z_1, \dots, z_s) \in U \mid z_{s_\tau+1} = \cdots = z_l = 0, z_{l+1} = \cdots = z_s = 1\}$$

$$G_\tau := \{{}^t(t_1, \dots, t_{s+r}) \in G \subset (\mathbb{C}^*)^{s+r} \mid t_{l+1} = \cdots = t_s = 1\}.$$

For a cone $\sigma \in \Delta$ satisfying $\tau \not\subset \sigma$, we have $U_\sigma \cap V_\tau = \emptyset$, which implies that

$$V_\tau = (\bigcup_{\tau \subset \sigma \in \Delta} U_\sigma) \cap \{z_{s_\tau+1} = \cdots = z_l = 0, z_{l+1} = \cdots = z_s = 1\}.$$

Hence we have a natural isomorphism $U_{\Sigma_\tau} \cong V_\tau$ induced by the inclusion

$$\mathbb{C}^{\Delta_\tau(1)} = \mathbb{C}^{s_\tau} \hookrightarrow \mathbb{C}^{\Delta(1)} = \mathbb{C}^s \quad (z_1, \dots, z_{s_\tau}) \mapsto (z_1, \dots, z_{s_\tau}, \overbrace{0, \dots, 0}^{l-s_\tau}, \overbrace{1 \dots, 1}^{s-l}).$$

Recall the definitions in §2.2 of

$$A = (\boldsymbol{a}_1 | \dots | \boldsymbol{a}_r) \in M(n+r, r, \mathbb{Z}) \text{ and } B = (\boldsymbol{b}_1 | \dots | \boldsymbol{b}_s) \in M(n+r, s, \mathbb{Z}),$$

where $\boldsymbol{a}_i, \boldsymbol{b}_i$ are column vectors in $\mathbb{Z}^{n+r}$. Define

$$A_\tau = (\boldsymbol{b}_{s_\tau+1} | \dots | \boldsymbol{b}_l | \boldsymbol{a}_1 | \dots | \boldsymbol{a}_r) \in M(n+r, r+l-s_\tau, \mathbb{Z})$$

and

$$B = (\boldsymbol{b}_1 | \dots | \boldsymbol{b}_{s_\tau}) \in M(n+r, s_\tau, \mathbb{Z}).$$

Then the short exact sequence

$$0 \to \mathbb{Z}^{r+l-s_\tau} \stackrel{A_\tau}{\to} \mathbb{Z}^{n+r} \to N(\tau) \to 0$$

gives a projective resolution of $N(\tau)$, and B_τ defines the map β_τ as B defines β. By the explicit description (9), we have an isomorphism $G_{\Sigma_\tau} \cong G_\tau$. Then the isomorphism between U_{Σ_τ} and V_τ becomes $G_{\Sigma_\tau} \cong G_\tau$-equivariant, and hence we get an isomorphism

$$\mathcal{X}_{\Sigma_\tau} (= [U_{\Sigma_\tau}/G_{\Sigma_\tau}]) \cong [V_\tau/G_\tau]$$

of stacks. Now put

$$V(\tau) := \{z_{s_\tau+1} = \cdots = z_l = 0 \text{ in } U\} \; (\subset \bigcup_{\tau \subset \sigma \in \Delta} U_\sigma).$$

Then we have the following lemma.

Lemma 2.3. *We have an isomorphism $[V_\tau/G_\tau] \cong [V(\tau)/G]$ of stacks. In particular, there is a closed embedding*

$$\iota \colon \mathcal{X}_{\Sigma_\tau} \cong [V(\tau)/G] \hookrightarrow \mathcal{X} = [U/G].$$

Proof. Embeddings $V_\tau \subset V(\tau)$, $G_\tau \subset G$ gives a morphism $\varphi \colon [V_\tau/G_\tau] \to [V(\tau)/G]$ of stacks. We show that φ is an isomorphism.

To show that φ is essentially surjective, since $[V(\tau)/G]$ is a sheaf, it is enough to show that for any object P of $[V(\tau)/G](W)$ over a scheme W there exists an étale covering $\{W_i \to W\}_i$ of W such that $P|_{W_i}$ is in the essential image of $\varphi_{W_i} \colon [V_\tau/G_\tau](W_i) \to [V(\tau)/G](W_i)$ for any i. First take an étale covering such that $P|_{W_i}$ is given by a trivial principal G-bundle $W_i \times G \to W_i$ and a G-equivariant morphism ψ:

$$V(\tau) \stackrel{\psi}{\longleftarrow} W_i \times G \longrightarrow W_i.$$

Furthermore for each i, we may assume that there exists a cone $\sigma \in \Delta$ satisfying $\tau \subset \sigma$, such that $\operatorname{im} \psi \subset U_\sigma \cap V(\tau)$.

Then by the explicit description (9) and (10), we can take an étale cover $W_i' \to W_i$ such that the morphism

$$W_i' \longrightarrow W_i \stackrel{\psi|_{W_i \times \mathrm{id}_G}}{\longrightarrow} U_\sigma \cap V(\tau)$$

is also decomposed as

$$W_i' \longrightarrow U_\sigma \cap V_\tau \hookrightarrow U_\sigma \cap V(\tau)$$

up to the G-action on U_σ. Hence $P|_{W_i'}$ belongs to the essential image of $\varphi_{W_i'}$.

Note that for the G-action on U the subgroup of elements of G keeping V_τ is equal to G_τ. Hence we can see that φ is fully faithful. This completes the proof. $\square$

We call $\mathcal{X}_{\Sigma_\tau}$ the *toric substack of $\mathcal{X}$ corresponding to the cone τ.* We often denote it by

$$[z_{s_\tau+1} = \cdots = z_l = 0].$$

For $\boldsymbol{k} = {}^t(k_1, \ldots, k_s) \in \mathbb{Z}^s$ and $\boldsymbol{l} = {}^t(l_1, \ldots, l_r) \in \mathbb{Z}^r$, we put

$$\boldsymbol{k}_\tau = {}^t(k_1, \ldots, k_{s_\tau}) \in \mathbb{Z}^{s_\tau}, \quad \boldsymbol{l}_\tau = {}^t(k_{s_\tau+1}, \ldots, k_l, l_1, \ldots, l_r) \in \mathbb{Z}^{l-s_\tau+r}$$

and $\boldsymbol{D}_\tau = {}^t(\iota^*\mathcal{D}_1, \ldots, \iota^*\mathcal{D}_{s_\tau})$. By Lemma 2.3, we have

$$\iota_*\mathcal{O}_{\mathcal{X}_{\Sigma_\tau}}(\boldsymbol{k}_\tau \boldsymbol{D}_\tau)_{\boldsymbol{l}_\tau} = \mathcal{O}_{\mathcal{X}}(\boldsymbol{k}\boldsymbol{D})_{\boldsymbol{l}} \otimes \iota_*\mathcal{O}_{\mathcal{X}_{\Sigma_\tau}}, \tag{11}$$

since we can compute push-forward by the embedding $[V(\tau)/G] \hookrightarrow [U/G]$ as in the proof of [BH1, Theorem 9.1].

2.4 Morphisms between toric DM stacks

In this subsection we consider a morphism between toric DM stacks $\mathcal{X}$ and $\mathcal{X}'$.

First let us consider the following morphism of triangles of the derived category of $\mathbb{Z}$-modules:

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}^s \arrow[r, "\beta"] \arrow[d, "\gamma_1 = C"'] & N \arrow[r] \arrow[d, "\gamma_2"] & \mathrm{Cone}(\beta) \arrow[r] \arrow[d, "\gamma_3"] & \cdots \\ \cdots \arrow[r] & \mathbb{Z}^{s'} \arrow[r, "\beta'"] & N' \arrow[r] & \mathrm{Cone}(\beta') \arrow[r] & \cdots, \end{tikzcd} \tag{12}$$

where γ_1 is a matrix $C = (c_{i,j}) \in M(s', s, \mathbb{Z})$. We assume that

(T1) γ_2 induces a map $\bar{\gamma}_2 \colon (\bar{N}, \Delta) \to (\bar{N}', \Delta')$ of fans, that is, for every cone $\sigma \in \Delta$, there exists a cone $\sigma' \in \Delta'$ satisfying $\gamma_{2,\mathbb{R}}(\sigma) \subset \sigma'$.

Take j satisfying $(\gamma_2 \circ \beta)_{\mathbb{R}}(\boldsymbol{f}_j) \in \sigma'$ for some cone $\sigma' \in \Delta'$. We furthermore assume that

(T2) we have

$$\gamma_1(\boldsymbol{f}_j) = \sum_{i=1}^{s'} c_{i,j}\boldsymbol{f}'_i \in \bigoplus_{\beta'_{\mathbb{R}}(\boldsymbol{f}'_i) \in \sigma'} \mathbb{Z}_{\geq 0}\boldsymbol{f}'_i.$$

As its consequence, we know that every $c_{i,j}$ is non-negative. Under these assumptions we construct a morphism $\varphi \colon \mathcal{X} = [U/G] \to \mathcal{X}' = [U'/G']$ of DM stacks as follows.

First consider the morphism

$$\gamma_{1,\mathbb{C}^*} := \gamma_1 \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}^*} \colon (\mathbb{C}^*)^s \to (\mathbb{C}^*)^{s'} \quad (z_j) \mapsto \left(z'_i\right) = \left(\prod_j z_j^{c_{i,j}}\right).$$

We can naturally extend it to a morphism $\mathbb{C}^s \to \mathbb{C}^{s'}$, which is also denoted by $\gamma_{1,\mathbb{C}^*}$. Then $\gamma_{1,\mathbb{C}^*}$ induces the morphism $U \to U'$ as follows: Take a point $p = (p_j) \in U_\sigma = \mathbb{C}^s \backslash \{z_\sigma = 0\}$ for some cone $\sigma \in \Delta$. If $p_j = 0$, then j satisfies that $\beta_{\mathbb{R}}(\boldsymbol{f}_j) \in \sigma$. Take a cone σ' such that $\gamma_{2,\mathbb{R}}(\sigma) \subset \sigma'$. Then the second assumption above implies that $c_{i,j} = 0$ for any i with $\beta'_{\mathbb{R}}(\boldsymbol{f}'_i) \notin \sigma'$. This means that $\gamma_{1,\mathbb{C}^*}(p) \in U'_{\sigma'}$. We again denote this morphism by

$$\gamma_{1,\mathbb{C}^*} \colon U \to U'.$$

On the other hand, the map γ_3 defines a group homomorphism $\rho = \mathrm{Hom}(H^1(\gamma_3^\star), \mathbb{C}^*))$

$$\rho\colon G = \mathrm{Hom}_{\mathbb{Z}}(H^1(\mathrm{Cone}(\beta)^\star), \mathbb{C}^*) \to G' = \mathrm{Hom}_{\mathbb{Z}}(H^1(\mathrm{Cone}(\beta')^\star), \mathbb{C}^*).$$

Then we have the following commutative diagram

$$\begin{tikzcd} G \times U \arrow[r] \arrow[d, "\rho\times\gamma_{1,\mathbb{C}^*}"'] & U \arrow[d, "\gamma_{1,\mathbb{C}^*}"] \\ G' \times U' \arrow[r] & U', \end{tikzcd} \tag{13}$$

where the horizontal arrows are defined by the actions of G on U and G' on U'. This diagram determines a morphism

$$\varphi\colon \mathcal{X} = [U/G] \to \mathcal{X}' = [U'/G']$$

of DM stacks.

Remark 2.4. *By [Iw2, Theorem 1.2], we see that giving torus equivariant morphisms between toric DM orbifolds is equivalent to giving homomorphism γ_1, γ_2 as in (12) satisfying assumptions (T1) and (T2).*

Let us take projective resolutions of N and N';

$$0 \to \mathbb{Z}^r \overset{A}{\to} \mathbb{Z}^{n+r} \to N \to 0 \text{ and } 0 \to \mathbb{Z}^{r'} \overset{A'}{\to} \mathbb{Z}^{n'+r'} \to N' \to 0$$

for $A \in M(n+r, r)$ and $A' \in M(n'+r', r')$. The map γ_2 is determined by matrices

$$D = (d_{i,j}) \in M(n'+r', n+r, \mathbb{Z}), \quad E = (e_{i,j}) \in M(r', r, \mathbb{Z}),$$

which make the diagram in the right square commutative (but not necessarily in the left):

$$\begin{tikzcd} \mathbb{Z}^s \arrow[r, "B"] \arrow[d, "C"'] & \mathbb{Z}^{n+r} \arrow[d, "D"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}^r \arrow[l, "A"'] \arrow[d, "E"] \\ \mathbb{Z}^{s'} \arrow[r, "B'"'] & \mathbb{Z}^{n'+r'} & \mathbb{Z}^{r'}. \arrow[l, "A'"] \end{tikzcd}$$

Here $B \in M(n+r, s)$ and $B' \in M(n'+r', s')$ give lifts of the maps $\beta\colon \mathbb{Z}^s \to N$ and $\beta'\colon \mathbb{Z}^{s'} \to N'$ respectively. By the commutativity of (12), we have a homotopy homomorphism $F = (f_{i,j})\colon \mathbb{Z}^s \to \mathbb{Z}^{r'}$ such that $DB - B'C = A'F$. The maps $\gamma_{1,\mathbb{C}^*}$ and ρ are described as follows:

$$\begin{aligned} &\gamma_{1,\mathbb{C}^*}\colon U \to U' \quad (z_j) \mapsto (z'_i) = \left(\prod_j z_j^{c_{i,j}}\right) \\ &\rho\colon G \to G' \quad (t_j) \mapsto (t'_i), \end{aligned} \tag{14}$$

where we put

$$t'_i := \prod_{j=1}^{s} t_j^{c_{i,j}} \text{ for } i = 1, \dots, s' \quad \text{ and } \quad t'_{s'+k} := \prod_{j=i}^{s} t_j^{f_{k,j}} \prod_{l=1}^{r} t_{s+l}^{e_{k,l}} \text{ for } k = 1, \dots, r',$$

and $z_j^{c_{i,j}} = 1$ when $z_j = c_{i,j} = 0$. Consequently we have

$$\varphi^*\mathcal{O}_{\mathcal{X}'}(\mathcal{D}'_i) = \mathcal{O}_{\mathcal{X}}(\sum_j c_{i,j}\mathcal{D}_j) \text{ and } \varphi^*\mathcal{O}_{\mathcal{X}',\boldsymbol{g}'^*_k} = \mathcal{O}_{\mathcal{X}}(\sum_j f_{k,j}\mathcal{D}_j)_{\sum_l e_{k,l}\boldsymbol{g}^*_l}. \tag{15}$$

For the homogeneous coordinate ring $S = \mathbb{C}[z_1, \dots, z_s]$ (resp. $S' = \mathbb{C}[z'_1, \dots, z'_{s'}]$) of $\mathcal{X}$ (resp. $\mathcal{X}'$), we define the map

$$\varphi^\sharp\colon S' \to S \quad (z'_i) \mapsto \prod_j z_j^{c_{i,j}}$$

so that $\varphi^\sharp(z'^{\boldsymbol{k}'}) = z^{\gamma_1^*(\boldsymbol{k}')}$. In particular, we have $\varphi^\sharp(S'_{\chi'}) \subset S_{\rho^\vee(\chi')}$, where

$$\rho^\vee\colon G'^\vee \to G^\vee$$

is the $\mathbb{C}^*$-dual map $\mathrm{Hom}_{\mathbb{Z}}(\rho, \mathbb{C}^*)$ of ρ. We have the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z}) \arrow[r] & \mathbb{Z}^s \arrow[r, "\mathrm{cl}"] & \mathrm{DG}(\beta) = G^\vee \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(N,\mathbb{Z}) \arrow[r] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(N',\mathbb{Z}) \arrow[r] \arrow[u, "\gamma_2^*"] & \mathbb{Z}^{s'} \arrow[r, "\mathrm{cl}"] \arrow[u, "\gamma_1^* = {}^tC"'] & \mathrm{DG}(\beta') = G'^\vee \arrow[r] \arrow[u, "\rho^\vee"'] & \mathrm{Ext}^1_{\mathbb{Z}}(N',\mathbb{Z}) \arrow[r] \arrow[u, "\gamma_2^*[1]"] & \cdots . \end{tikzcd} \tag{16}$$

If a $G'^\vee$-graded S'-module $M = \bigoplus_{\chi' \in G'^\vee} M_{\chi'}$ gives a coherent sheaf $\mathcal{F}$ on $\mathcal{X}'$ by (4), then a $G^\vee$-graded S-module $M \otimes_{S'} S$ corresponds to the pull-back $\varphi^*\mathcal{F}$, where the grading is given by

$$(M \otimes_{S'} S)_\chi = \sum_{\eta' \in G'^\vee} M_{\eta'} \otimes_{S'} S_{\chi - \rho^\vee(\eta')}$$

for each $\chi \in G^\vee$.

On the other hand, a $G^\vee$-graded S-module $L = \bigoplus_{\chi \in G^\vee} L_\chi$ gives a coherent sheaf $\mathcal{G}$ on $\mathcal{X}$, and the sheaf $\varphi_*\mathcal{G}$ corresponds a $G'^\vee$-graded S'-module ${}_{S'}N$ whose grading is given by

$$({}_{S'}L)_{\chi'} = L_{\rho^\vee(\chi')}$$

for $\chi' \in G'^\vee$. Here S'-module structure of ${}_{S'}N$ is given by $\varphi^\sharp$. This S'-module structure is compatible with the $G'^\vee$-grading of ${}_{S'}L$, since $\varphi^\sharp(S'_{\chi'}) \subset S_{\rho^\vee(\chi')}$.

The functor ${}_{S'}(-)\colon \mathrm{gr}S/\operatorname{tor}S \to \mathrm{gr}S'/\operatorname{tor}S'$ is the right adjoint functor of $(-)\otimes_S S'$. Since $\varphi_*\colon \operatorname{Coh}\mathcal{X} \to \operatorname{Coh}\mathcal{X}'$ is the right adjoint functor of $\varphi^*\colon \operatorname{Coh}\mathcal{X}' \to \operatorname{Coh}\mathcal{X}$, both functors ${}_{S'}(-)$ and φ_* must coincide by the correspondence in (4). Hence for any $\mathcal{G} \in \operatorname{Coh}\mathcal{X}$, we have a natural isomorphism

$$H^0(U', \varphi_*\mathcal{G}) \cong \bigoplus_{\chi' \in G'^\vee} H^0(U, \mathcal{G})_{\rho^\vee(\chi')} \tag{17}$$

in $\mathrm{gr}S'/\operatorname{tor}S'$.

Lemma 2.5. *Take a morphism $\varphi\colon \mathcal{X} \to \mathcal{X}'$ determined by matrices $C = (c_{i,j}) \in M(s', s, \mathbb{Z}), D \in M(n'+r', n+r, \mathbb{Z})$ and $E \in M(r', r, \mathbb{Z})$ as above. Then we have the following.*

(i) Suppose that for each i there exits j such that $c_{i,j} > 0$, and entries of the j-th column vector of C are all zero except $c_{i,j}$. Furthermore, in the diagram (16), we assume that γ_2^* is surjective, and $\gamma_2^*[1]$ is injective. Then we have $\varphi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}'}$.

(ii) Assume that $s = s'$, $C = (c_i\delta_{i,j})$ for $c_i \in \mathbb{Z}_{>0}$, and that the map of fans $\bar{\gamma}_2\colon (\bar{N}, \Delta) \to (\bar{N}', \Delta')$ is an isomorphism. Then we have $\mathbb{R}^i\varphi_*\mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ and $i > 0$.

Proof. (i) For any $\chi' \in G'^\vee$, we consider sets

$$\mathfrak{S}' = \{\boldsymbol{k}' \in \mathbb{Z}^{s'}_{\geq 0} \mid \mathrm{cl}(\boldsymbol{k}') = \chi'\}, \quad \mathfrak{S} = \{\boldsymbol{k} \in \mathbb{Z}^{s}_{\geq 0} \mid \mathrm{cl}(\boldsymbol{k}) = \rho^\vee(\chi')\}.$$

By the assumption on C, we see that ${}^tC\colon \mathbb{Z}^{s'} \to \mathbb{Z}^s$ is injective, and $({}^tC)^{-1}(\mathbb{Z}^s_{\geq 0}) = \mathbb{Z}^{s'}_{\geq 0}$. When $\mathfrak{S}' \neq \emptyset$, in the diagram (16), combining the surjectivity of γ_2^* and the injectivity of ${}^tC$, we conclude that the map ${}^tC$ induces a bijection between $\mathfrak{S}'$ and $\mathfrak{S}$.

On the other hand, the injectivity of the map $\gamma_2^*[1]$ is equivalent to the condition ${\rho^\vee}^{-1}(\mathrm{cl}(\mathbb{Z}^s)) = \mathrm{cl}(\mathbb{Z}^{s'})$. So if $\mathfrak{S}' = \emptyset$, then $\mathfrak{S} = \emptyset$. Hence we obtain an isomorphism $S' \cong {}_{S'}S$ by the map $\varphi^\sharp$. This completes the proof.

(ii) For any $\sigma \in \Delta' = \Delta$, we have $\varphi^\sharp(z'_\sigma) = z^{\boldsymbol{k}} z_\sigma$ for some $\boldsymbol{k} \in \mathbb{Z}^s_{\geq 0}$, where $\boldsymbol{k} = (k_i)$ satisfies that $k_i = 0$ if $\beta_{\mathbb{R}}(\boldsymbol{f}_i) \in \sigma$. Hence if we take a $G^\vee$-graded S-module $L \in \operatorname{tor}S$, then ${}_{S'}L$ also belongs

to $\operatorname{tor} S'$. We see that an exact sequence in the abelian category $\mathrm{gr}S/\operatorname{tor} S$ is given by an exact sequence of graded S-modules

$$0 \to L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3$$

such that $\operatorname{coker} g$ belongs to $\operatorname{tor} S$. Hence $\varphi_*\colon \operatorname{Coh}\mathcal{X} \to \operatorname{Coh}\mathcal{X}'$ is an exact functor, which implies the assertion.

$\square$

3 Toric DM stacks vs. toric DM orbifolds

Let us introduce two kinds of *root stacks*, important notions in this note. In this section, we consider the toric DM stack $\mathcal{X} = \mathcal{X}_\Sigma$ associated to a stacky fan $\Sigma = (\beta, \Delta)$ in

$$N := \mathbb{Z}^n \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}$$

for some $a_i \in \mathbb{Z}_{>0}$. As in §2.2 we take matrices $A = \begin{pmatrix} O_{n\times r} \\ \operatorname{diag}(a_1, \ldots, a_r) \end{pmatrix} \in M(n+r, r, \mathbb{Z})$ and $B = (b_{i,j}) \in M(n+r, s, \mathbb{Z})$ such that $N \cong \operatorname{coker} A$ and β is defined by $\beta\colon \mathbb{Z}^s \xrightarrow{B} \mathbb{Z}^{n+r} \twoheadrightarrow \operatorname{coker} A$.

3.1 Root stacks of line bundles on toric DM stacks

For $r' > r$, $\boldsymbol{e} = {}^t(e_1, \ldots, e_{r'-r}) \in \mathbb{Z}_{>0}^{r'-r}$ and a collection $\boldsymbol{L} = {}^t(\mathcal{L}_1, \ldots, \mathcal{L}_{r'-r})$ of line bundles on $\mathcal{X}$, we consider the ($\boldsymbol{e}$-th) *root stack* $\sqrt[\boldsymbol{e}]{\boldsymbol{L}/\mathcal{X}}$ of $(\mathcal{X}, \boldsymbol{L})$ ([FMN, 1.3.a]), that is, the fiber product:

$$\begin{tikzcd} \sqrt[\boldsymbol{e}]{\boldsymbol{L}/\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathcal{B}\mathbb{C}^*)^{r'-r} \arrow[d, "\wedge \boldsymbol{e}"] \\ \mathcal{X} \arrow[r, "\boldsymbol{L}"] & (\mathcal{B}\mathbb{C}^*)^{r'-r}. \end{tikzcd}$$

Here $\mathcal{B}\mathbb{C}^*$ is the quotient stack of a point by the trivial $\mathbb{C}^*$-action. By abuse of notation, we use symbols $\boldsymbol{L}\colon \mathcal{X} \to (\mathcal{B}\mathbb{C}^*)^{r'-r}$ and $\wedge\boldsymbol{e}\colon (\mathcal{B}\mathbb{C}^*)^{r'-r} \to (\mathcal{B}\mathbb{C}^*)^{r'-r}$ to denote the morphisms induced by $\boldsymbol{L}$ and $\wedge\boldsymbol{e}\colon (\mathbb{C}^*)^{r'-r} \to (\mathbb{C}^*)^{r'-r}$ respectively.

The root stack $\sqrt[\boldsymbol{e}]{\boldsymbol{L}/\mathcal{X}}$ is constructed as a toric stack in the following way. We take $(\boldsymbol{k}_i, \boldsymbol{l}_i) \in \mathbb{Z}^s \oplus \mathbb{Z}^r$ such that $\mathcal{L}_i = \mathcal{O}_{\mathcal{X}}(\boldsymbol{k}_i\boldsymbol{D})_{\boldsymbol{l}_i}$ in $\operatorname{Pic}\mathcal{X}$ for $i = 1, \ldots, r'-r$ and matrices

$$A' = \begin{pmatrix} A & 0 & \ldots & 0 \\ -{}^t\boldsymbol{l}_1 & e_1 & & \\ \vdots & & \ddots & \\ -{}^t\boldsymbol{l}_{r'-r} & & & e_{r'-r} \end{pmatrix} \in M(n+r', r', \boldsymbol{Z}), \quad B' = \begin{pmatrix} B \\ -{}^t\boldsymbol{k}_1 \\ \vdots \\ -{}^t\boldsymbol{k}_{r'-r} \end{pmatrix} \in M(n+r', s, \mathbb{Z}).$$

We define an abelian group $N' = \operatorname{coker} A'$ and a stacky fan $\Sigma' = (\beta', \Delta)$ in N' by the map $\beta' \colon \mathbb{Z}^s \xrightarrow{B'} \mathbb{Z}^{n+r'} \twoheadrightarrow \operatorname{coker}(A')$.

We have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^s \arrow[r, "B'"] \arrow[d, equal, "C=\mathrm{id}_{\mathbb{Z}^s}"'] & \mathbb{Z}^{n+r'} \arrow[d, equal, "D"] & \mathbb{Z}^{r'} \arrow[l, "A'"'] \arrow[d, "E"] \\
\mathbb{Z}^s \arrow[r, "B"'] & \mathbb{Z}^{n+r} & \mathbb{Z}^r, \arrow[l, "A"]
\end{tikzcd}$$

where D and E is the projections. Then we have a morphism $\Psi \colon \mathcal{X}' = \mathcal{X}_{\Sigma'} \to \mathcal{X}$ by the result in §2.4, and Ψ satisfies that

$$\Psi^* \mathcal{O}_{\mathcal{X}}(\boldsymbol{k}\boldsymbol{D})_{\boldsymbol{l}} = \mathcal{O}_{\mathcal{X}'}(\boldsymbol{k}\boldsymbol{D}')_{\boldsymbol{l}}$$

for any $(\boldsymbol{k}, \boldsymbol{l}) \in \mathbb{Z}^s \oplus \mathbb{Z}^r$ by (15), where we identify $\boldsymbol{l}$ with the element in $\mathbb{Z}^{r'}$ by the injection $\mathbb{Z}^r \to \mathbb{Z}^r \oplus \mathbb{Z}^{r'-r} \cong \mathbb{Z}^{r'}$.

In particular, by the choice of A' and B', we have isomorphisms

$$\Psi^* \mathcal{O}_{\mathcal{X}}(\boldsymbol{k}_i \boldsymbol{D})_{\boldsymbol{l}_i} \cong (\mathcal{O}_{\mathcal{X}', \boldsymbol{g}'^*_{r+i}})^{\otimes e_i}$$

for $i = 1, \dots, r' - r$. Hence Ψ and

$$\boldsymbol{L}' = {}^t(\mathcal{O}_{\mathcal{X}', \boldsymbol{g}'^*_{r+1}}, \dots, \mathcal{O}_{\mathcal{X}', \boldsymbol{g}'^*_{r'}}) \colon \mathcal{X}' \to (\mathcal{B}\mathbb{C}^*)^{r'-r}$$

gives a morphism $\mathcal{X}' \to \sqrt[\boldsymbol{e}]{\boldsymbol{L}/\mathcal{X}}$. By [Pe, Theorem 2.6], we see that this is an isomorphism.

We have the following theorem.

Theorem 3.1. *For $\mathcal{X}' = \sqrt[\boldsymbol{e}]{\boldsymbol{L}/\mathcal{X}}$, we have an equivalence*

$$\operatorname{Coh} \mathcal{X}' \cong \bigoplus_{\substack{(l_i) \in \mathbb{Z}^{r'-r} \\ 0 \le l_i < e_i}} (\Psi^* \operatorname{Coh} \mathcal{X}) \otimes \mathcal{O}_{\mathcal{X}', \sum_i l_i \boldsymbol{g}'^*_{r+i}}.$$

Proof. Applying [IU, Lemma 4.1] repeatedly, we obtain the assertion. ☐

3.2 The rigidification of toric DM stacks

We define the *rigidification* $\mathcal{X}^{\mathrm{rig}}$ of $\mathcal{X}$ according to [FMN] as follows. Let us take a stacky fan $\Sigma^{\mathrm{rig}} = (\beta^{\mathrm{rig}}, \Delta)$ in $\bar{N} = \mathbb{Z}^n$, where we define $\beta^{\mathrm{rig}} \colon \mathbb{Z}^s \to \bar{N}$ to be a composition of β and a natural surjection $N \to \bar{N}$, and consider the toric DM orbifold $\mathcal{X}^{\mathrm{rig}} := \mathcal{X}_{\Sigma^{\mathrm{rig}}}$.

We write $B = (b_{i,j})$ for $b_{i,j} \in \mathbb{Z}$ and put $\mathcal{L}_i = \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}(-\sum_j b_{n+i,j} \mathcal{D}_j)$ for $i = 1, \dots, r$. Then the stack $\mathcal{X}$ is isomorphic to the ${}^t(a_1, \dots, a_r)$-th root stack of $(\mathcal{X}^{\mathrm{rig}}, (\mathcal{L}_1, \dots, \mathcal{L}_r))$. In particular, we

have a morphism $\Psi\colon \mathcal{X} \to \mathcal{X}^{\mathrm{rig}}$ as in §3.1. We call this Ψ the *rigidification morphism*. By applying Theorem 3.1 to Ψ, we obtain the following corollaries:

Corollary 3.2. *Let $\mathcal{X}$ be a toric DM stack and $\mathcal{X}^{rig}$ its rigidification. Then we have an equivalence*

$$\operatorname{Coh}\mathcal{X} \cong \bigoplus_{\substack{(l_i)\in\mathbb{Z}^r\\ 0\le l_i<a_i}} (\Psi^* \operatorname{Coh}\mathcal{X}^{\mathrm{rig}}) \otimes \mathcal{O}_{\mathcal{X},\sum_i l_i \boldsymbol{g}_i^*}.$$

The following must be well-known to specialists. It is used in the proof of Lemma 4.4.

Corollary 3.3. *Let $\mathcal{X}$ be a toric DM stack. Then* $\operatorname{Coh}\mathcal{X}$ *has a finite homological dimension, namely any coherent sheaf on $\mathcal{X}$ has a locally free resolution of finite length.*

Proof. $\operatorname{Coh}\mathcal{X}^{\mathrm{rig}}$ has a finite homological dimension by the proof of [BH1, Theorem 4.6]. Hence the assertion follows from Corollary 3.2. □

The following are stacky generalizations of the results for toric DM orbifolds in [Ka] and [BHu, Theorem 7.3].

Corollary 3.4. *(i) Let $\mathcal{X}$ be a toric DM stack. Then $D^b(\mathcal{X})$ has a full exceptional collection consisting of coherent sheaves.*

(ii) Let $\mathcal{X}$ be a two dimensional toric DM stack whose rigidification $\mathcal{X}^{rig}$ has the ample anti-canonical divisor. Then $D^b(\mathcal{X})$ has a full strong exceptional collection consisting of line bundles.

Proof. The assertion (i) (respectively (ii)) directly follows from Corollary 3.2 and the result in [Ka] (resp. [BHu, Theorem 7.3]). □

3.3 Root stacks of effective Cartier divisors on toric DM stacks

For positive integers $c_1, \dots, c_s$, we define another stacky fan $\Sigma'' = (\beta'', \Delta)$ in N by replacing β in Σ with

$$\beta''\colon \mathbb{Z}^s \overset{BC}{\to} \mathbb{Z}^{n+r} \twoheadrightarrow \operatorname{coker} A,$$

where we put $C = \operatorname{diag}(c_1, \dots, c_s) \in M(s, s, \mathbb{Z})$. We call $\mathcal{X}'' = \mathcal{X}_{\Sigma''}$ the ($\boldsymbol{c}$-th) *root stack* of $(\mathcal{X}, \boldsymbol{D})$ ([FMN, 1.3.b]) and denote it by

$$\sqrt[\boldsymbol{c}]{\boldsymbol{D}/\mathcal{X}},$$

where we define $\boldsymbol{c} = {}^t(c_1, \dots, c_s)$ and $\boldsymbol{D} = {}^t(\mathcal{D}_1, \dots, \mathcal{D}_s)$ is the collection of toric divisors on $\mathcal{X}$ as in §2.2. We have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}^s \arrow[r, "BC"] \arrow[d, "C"'] & \mathbb{Z}^{n+r} \arrow[d, equal, "D=\mathrm{id}_{\mathbb{Z}^{n+r}}"] & \mathbb{Z}^r \arrow[l, "A"'] \arrow[d, equal, "E=\mathrm{id}_{\mathbb{Z}^r}"] \\ \mathbb{Z}^s \arrow[r, "B"'] & \mathbb{Z}^{n+r} & \mathbb{Z}^r. \arrow[l, "A"] \end{tikzcd}$$

Then we have a morphism $\varphi\colon \mathcal{X}'' = \sqrt[\boldsymbol{c}]{\boldsymbol{D}/\mathcal{X}} \to \mathcal{X}$ by the result in §2.4. We call φ the *root construction morphism.* We obtain the following by Lemma 2.5 (i), (ii).

Lemma 3.5. $\mathbb{R}\varphi_* \mathcal{O}_{\mathcal{X}''} = \mathcal{O}_{\mathcal{X}}$.

Next assume furthermore that $\mathcal{X} = \mathcal{X}_\Sigma$ is an orbifold. Then in the notation in §2.1, $\mathcal{X}$ is denoted by $\mathcal{X}\big(X, \sum_{i=1}^s b_i D_i\big)$ for some $b_i \in \mathbb{Z}_{>0}$ and the coarse moduli space $X = X_\Delta$ of $\mathcal{X}$. Then $\mathcal{X}$ is realized as a root stack $\sqrt[\boldsymbol{b}]{\boldsymbol{D}/\mathcal{X}^{\mathrm{can}}}$ over the canonical stack $\mathcal{X}^{\mathrm{can}} = \mathcal{X}\big(X, \sum D_i\big)$ for $\boldsymbol{b} = {}^t(b_1, \dots, b_s)$. Additionally if X is smooth, then $\mathcal{X}$ is a root stack $\sqrt[\boldsymbol{b}]{\boldsymbol{D}/X}$ over X by Remark 2.2.

4 Frobenius push-forward for toric DM stacks

4.1 Generators

Let us begin this section with important definitions. Suppose that $\mathcal{X}$ is a smooth complete DM stack.

Definition 4.1. *For a subset $\mathcal{S} \subset D^b(\mathcal{X})$, $\langle \mathcal{S} \rangle$ denotes the smallest full triangulated subcategory containing $\mathcal{S}$ of $D^b(\mathcal{X})$ such that $\langle \mathcal{S} \rangle$ is stable under taking direct summands and direct sums.*

We say that $\mathcal{S}$ is a generator *of $D^b(\mathcal{X})$, or $\mathcal{S}$* generates *$D^b(\mathcal{X})$ if $\langle \mathcal{S} \rangle = D^b(X)$. If $\mathcal{S}$ consists of a single element α, we just say that α is a generator of $D^b(\mathcal{X})$.*

Definition 4.2. *(i) An object $\mathcal{E} \in D^b(\mathcal{X})$ is called* exceptional *if it satisfies*

$$\mathrm{Hom}^i_{D^b(\mathcal{X})}(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) An ordered set $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ of exceptional objects is called an exceptional collection *if the following condition holds;*

$$\mathrm{Hom}^i_{D^b(\mathcal{X})}(\mathcal{E}_k, \mathcal{E}_j) = 0$$

for all $k > j$ and all i. When we say that a finite set $\mathcal{S}$ of objects is an exceptional collection, it means that $\mathcal{S}$ forms an exceptional collection in an appropriate order.

(iii) An exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ *is called* strong *if*

$$\operatorname{Hom}^i_{D^b(\mathcal{X})}(\mathcal{E}_k, \mathcal{E}_j) = 0$$

for all k, j *and* $i \neq 0$.

(iv) An exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ *is called* full *if the set* $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ *generates* $D^b(\mathcal{X})$.

Remark 4.3. *Let* $\mathcal{X} = \mathcal{X}_\Sigma$ *be a toric DM orbifold associated with a stacky fan* $\Sigma = (\Delta, \beta)$. *If the toric DM stack* $\mathcal{X} = \mathcal{X}_\Sigma$ *has a full exceptional collection consisting of* n *exceptional objects, the rank of its Grothendieck group* $K(\mathcal{X})$ *is* n. *Furthermore suppose that* $-K_{\mathcal{X}}$ *is nef, which is equivalent to the condition that all* $\beta(\boldsymbol{f}_i)$*'s lie on the boundary of the convex hull of all* $\beta(\boldsymbol{f}_i)$*'s (see also §8 for the notion* nef*). Then it is proved in [BH2, Corollary 2.2 and Proposition 3.1 (iii)] that*

$$\operatorname{rk} K(\mathcal{X}) = \operatorname{rk} N! \operatorname{vol} \Delta,$$

where $\operatorname{vol} \Delta$ *is the volume of the convex hull of all* $\beta(\boldsymbol{f}_i)$*'s.*

4.2 Frobenius morphism

Below we use the terminology in §2.1 and 2.2. For a positive integer m, we consider the Frobenius morphism $F(= F_m) \colon \mathcal{X} \to \mathcal{X}$ induced by

$$\wedge m \colon U \to U \text{ and } \wedge m \colon G \to G.$$

Take both of $\gamma_1 \colon \mathbb{Z}^s \to \mathbb{Z}^s$ and $\gamma_2 \colon N \to N$ in (12) as the multiplication maps by m. Then we obtain the *Frobenius morphism* F, which is actually a generalization of the usual Frobenius morphism.

Lemma 4.4. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be toric DM stacks. Consider a proper morphism* $\varphi \colon \mathcal{X} \to \mathcal{Y}$ *which satisfies* $\mathbb{R}\varphi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$. *Then* $D^b(\mathcal{X}) = \langle F_* \mathcal{O}_{\mathcal{X}} \rangle$ *implies* $D^b(\mathcal{Y}) = \langle F_* \mathcal{O}_{\mathcal{Y}} \rangle$.

Proof. We see that $\operatorname{Coh} \mathcal{Y}$ has a finite homological dimension by Corollary 3.3. For any object $\mathcal{F} \in D^b(\mathcal{Y})$, by the assumption we have $\mathbb{L}\varphi^* \mathcal{F} \in \langle F_* \mathcal{O}_{\mathcal{X}} \rangle$. Since we have $\mathbb{R}\varphi_* F_* \mathcal{O}_{\mathcal{X}} = F_* \mathbb{R}\varphi_* \mathcal{O}_{\mathcal{X}} = F_* \mathcal{O}_{\mathcal{Y}}$, we see that $\mathcal{F} \cong \mathbb{R}\varphi_* \mathbb{L}\varphi^* \mathcal{F}$ belongs to $\langle F_* \mathcal{O}_{\mathcal{Y}} \rangle$. $\square$

4.3 Direct summands of Frobenius push-forward

We consider the Frobenius morphism

$$F \colon \mathcal{X} \to \mathcal{X}'.$$

Although the target space $\mathcal{X}'$ is $\mathcal{X}$ itself, we use the notation $\mathcal{X}'$ so that we distinguish the domain $\mathcal{X}$ and the target $\mathcal{X}'$. Theorem 4.5 for toric DM stacks generalizes the result for toric varieties in [Ac]. A similar proof in [Ac] works for the stacky case, but we give another proof using (17).

Theorem 4.5. *For $\chi \in G^\vee$ and $\chi' \in G'^\vee$, put*

$$m(\chi, \chi') := \sharp\{\boldsymbol{j} \in [0, m-1]^s \mid m\chi' = \chi - \mathrm{cl}(\boldsymbol{j})\}.$$

Then we have

$$F_* \mathcal{L}_\chi = \bigoplus_{\chi' \in G'^\vee} \mathcal{L}_{\chi'}^{\oplus m(\chi, \chi')}.$$

Proof. We identify $\operatorname{Coh} \mathcal{X}$ and $\operatorname{Coh}_G U$ by (2). We consider the graded S-module

$$H^0(U, \mathcal{L}_\chi) = \bigoplus_{\eta \in G^\vee} H^0(U, \mathcal{L}_\chi)_\eta$$

and recall that

$$H^0(U, \mathcal{L}_\chi)_\eta \cong \bigoplus_{\substack{\boldsymbol{i} \in \mathbb{Z}_{\geq 0}^s \\ \mathrm{cl}(\boldsymbol{i}) = \chi + \eta}} \mathbb{C} z^{\boldsymbol{i}},$$

since the G-actions in (5) and (10) induce the action on $H^0(U, \mathcal{L}_\chi)$ given by $z^{\boldsymbol{i}} \mapsto \chi_{\bar{\boldsymbol{i}}} \chi^{-1} z^{\boldsymbol{i}}$. We define a $G'^\vee$-graded S'-module $M = \bigoplus_{\eta' \in G'^\vee} M_{\eta'}$ by

$$M_{\eta'} := H^0(U, \mathcal{L}_\chi)_{m\eta'} \cong \bigoplus_{\substack{\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^s \\ \mathrm{cl}(\boldsymbol{k}) = \chi + m\eta'}} \mathbb{C} z^{\boldsymbol{k}}.$$

By (17), the S'-module M corresponds to $F_* \mathcal{L}_\chi$ by the equivalence (4).

We have a one-to-one correspondence between sets $\{\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^s \mid \mathrm{cl}(\boldsymbol{k}) = \chi + m\eta'\}$ and

$$\{(\boldsymbol{i}, \boldsymbol{j}) \in \mathbb{Z}_{\geq 0}^s \times [0, m-1]^s \mid m(\mathrm{cl}(\boldsymbol{i}) - \eta') = \chi - \mathrm{cl}(\boldsymbol{j})\}$$

given by $\boldsymbol{k} = m\boldsymbol{i} + \boldsymbol{j}$ such that $\boldsymbol{i} = \lfloor \frac{\boldsymbol{k}}{m} \rfloor$ and $\boldsymbol{j} \in [0, m-1]^s$. Thus for each $\eta' \in G'^\vee$, we have isomorphisms between components of $G'^\vee$-graded S'-modules

$$M_{\eta'} \cong \bigoplus_{\boldsymbol{j} \in [0, m-1]^s} \left(\bigoplus_{\substack{\boldsymbol{i} \in \mathbb{Z}_{\geq 0}^s \\ m(\mathrm{cl}(\boldsymbol{i}) - \eta') = \chi - \mathrm{cl}(\boldsymbol{j})}} \mathbb{C} z^{m\boldsymbol{i} + \boldsymbol{j}} \right) \cong \bigoplus_{\chi' \in G'^\vee} \left(\bigoplus_{\substack{\boldsymbol{j} \in [0, m-1]^s \\ m\chi' = \chi - \mathrm{cl}(\boldsymbol{j})}} \left(\bigoplus_{\substack{\boldsymbol{i} \in \mathbb{Z}_{\geq 0}^s \\ \mathrm{cl}(\boldsymbol{i}) = \chi' + \eta'}} \mathbb{C} z'^{\boldsymbol{i}} z^{\boldsymbol{j}} \right) \right).$$

To obtain the second isomorphism, we put $\chi' = \mathrm{cl}(\boldsymbol{i}) - \eta'$. The last component $\bigoplus_{\substack{\boldsymbol{i} \in \mathbb{Z}_{\geq 0}^s \\ \mathrm{cl}(\boldsymbol{i}) = \chi' + \eta'}} \mathbb{C} z'^{\boldsymbol{i}} z^{\boldsymbol{j}}$ is isomorphic to $H^0(U', \mathcal{L}_{\chi'})_{\eta'}$. Hence, summing up all $\eta' \in G'^\vee$, we have an isomorphism

$$M \cong \bigoplus_{\chi' \in G'^\vee} \bigoplus_{\substack{\boldsymbol{j} \in [0, m-1]^s \\ m\chi' = \chi - \mathrm{cl}(\boldsymbol{j})}} H^0(U', \mathcal{L}_{\chi'})$$

of S'-modules. Now the desired isomorphism

$$F_*\mathcal{L}_\chi \cong \bigoplus_{\chi' \in G'^\vee} \bigoplus_{\substack{\boldsymbol{j} \in [0,m-1]^s \\ m\chi' = \chi - \mathrm{cl}(\boldsymbol{j})}} \mathcal{L}_{\chi'} = \bigoplus_{\chi' \in G'^\vee} \mathcal{L}_{\chi'}^{\oplus m(\chi,\chi')}$$

follows from (4). $\square$

We denote by $\mathfrak{D}_{\mathcal{X}}(\mathcal{L}_\chi)$ the set of isomorphism classes of direct summands of $F_*\mathcal{L}_\chi$. For $\chi \in G^\vee$ and $\chi' \in G'^\vee$, there exist some $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{Z}^s$ and $\boldsymbol{l}, \boldsymbol{l}' \in \mathbb{Z}^r$ such that $\mathcal{L}_\chi = \mathcal{O}_{\mathcal{X}}(\boldsymbol{k}\boldsymbol{D})_{\boldsymbol{l}}$ and $\mathcal{L}_{\chi'} = \mathcal{O}_{\mathcal{X}'}(\boldsymbol{k}'\boldsymbol{D})_{\boldsymbol{l}'}$. Then we have $\chi = \chi_{\overline{\boldsymbol{k}+\boldsymbol{l}}}$ and $\chi' = \chi'_{\overline{\boldsymbol{k}'+\boldsymbol{l}'}}$ as in §2.2. If $m(\chi,\chi') > 0$ then we have

$$m\chi'_{\overline{\boldsymbol{k}'+\boldsymbol{l}'}} = \chi_{\overline{\boldsymbol{k}+\boldsymbol{l}}} - \mathrm{cl}(\boldsymbol{j})$$

for some $\boldsymbol{j} \in \mathbb{Z}^s \cap [0, m-1]^s$, and hence there exists an element $\boldsymbol{u} \in \mathbb{Z}^{n+r}$ such that we have

$$(\boldsymbol{k} + {}^tB\boldsymbol{u}) \oplus (\boldsymbol{l} + {}^tA\boldsymbol{u}) = (m\boldsymbol{k}' + \boldsymbol{j}) \oplus m\boldsymbol{l}' \quad \text{in } \mathbb{Z}^s \oplus \mathbb{Z}^r.$$

Thus we have $\boldsymbol{k}' = \lfloor \frac{\boldsymbol{k}+{}^tB\boldsymbol{u}}{m} \rfloor$ in $\mathbb{Z}^s$, $\boldsymbol{l}' = \frac{\boldsymbol{l}+{}^tA\boldsymbol{u}}{m}$ in $\mathbb{Z}^r$ and

$$\mathfrak{D}_{\mathcal{X}}(\mathcal{L}_\chi) = \left\{ \mathcal{O}_{\mathcal{X}}\left(\lfloor \frac{\boldsymbol{k}+{}^tB\boldsymbol{u}}{m} \rfloor \boldsymbol{D} \right)_{\frac{\boldsymbol{l}+{}^tA\boldsymbol{u}}{m}} \;\middle|\; \boldsymbol{u} \in \mathbb{Z}^{n+r} \cap [0, m-1]^{n+r}, \frac{\boldsymbol{l}+{}^tA\boldsymbol{u}}{m} \in \mathbb{Z}^r \right\}. \tag{18}$$

When N is torsion free, we have $r = 0$ and $N = \mathbb{Z}^n$. For each $i = 1, \ldots, s$, we take a primitive generator $\boldsymbol{v}_i \in N$ of ρ_i and positive integer b_i such that $\beta(\boldsymbol{f}_i) = b_i\boldsymbol{v}_i$. We have

$${}^tB\boldsymbol{u} = {}^t\left((\boldsymbol{u}, b_1\boldsymbol{v}_1), \ldots, (\boldsymbol{u}, b_s\boldsymbol{v}_s) \right),$$

where $(\boldsymbol{u}, b_i\boldsymbol{v}_i) \in \mathbb{R}$ is just the dot product on $\mathbb{R}^n$. Consequently, (18) is a generalization of Thomsen's result [Th] for smooth toric varieties.

The set $\mathfrak{D}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}})$ is stabilized for sufficiently divisible integers m in $F = F_m$, and denote this set by $\mathfrak{D}_{\mathcal{X}}$. By the above result, we have

$$\mathfrak{D}_{\mathcal{X}} = \left\{ \mathcal{O}_{\mathcal{X}}\left(\lfloor {}^tB\boldsymbol{u} \rfloor \boldsymbol{D} \right)_{{}^tA\boldsymbol{u}} \;\middle|\; \boldsymbol{u} \in [0,1)^{n+r}, {}^tA\boldsymbol{u} \in \mathbb{Z}^r \right\}. \tag{19}$$

4.4 Reduction to toric orbifolds

We reduce Main theorem to the orbifold's case. First take an arbitrary toric DM stack $\mathcal{X}$ and we use the notation in §3. Put $a := a_1 \cdots a_r$ and $\boldsymbol{a} := (a, \ldots, a) \in \mathbb{Z}^s$. For simplicity, denote the root

stack $\sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathcal{X}}$ by $\mathcal{X}' = \mathcal{X}_{\Sigma'}$ and consider the rigidification $\mathcal{X}'^{\mathrm{rig}}$ of $\mathcal{X}'$. Then β' is defined by the map

$$\mathbb{Z}^s \xrightarrow{\begin{pmatrix} aB_1 \\ \mathbf{0} \end{pmatrix}} \mathbb{Z}^{n+r} \twoheadrightarrow \operatorname{coker} A,$$

where $B_1 = (b_{i,j}) \in M(n, s, \mathbb{Z})$.

Here we identify toric divisors on $\mathcal{X}'$ and $\mathcal{X}'^{\mathrm{rig}}$, and denote by the same symbol $\boldsymbol{D}' = {}^t(\mathcal{D}'_1, \ldots, \mathcal{D}'_s)$. For the rigidification morphism $\Psi' \colon \mathcal{X}' \to \mathcal{X}'^{rig}$, by (15), we have $\Psi'^*\mathcal{O}_{\mathcal{X}'^{\mathrm{rig}}}(\boldsymbol{k}\boldsymbol{D}') \cong \mathcal{O}_{\mathcal{X}'}(\boldsymbol{k}\boldsymbol{D}')$ for any $\boldsymbol{k} \in \mathbb{Z}^s$. Furthermore by (19), we have

$$\begin{aligned}
\mathfrak{D}_{\mathcal{X}'} &= \left\{ \mathcal{O}_{\mathcal{X}'}\left(\lfloor a^t B_1 \boldsymbol{u}_1 \rfloor \boldsymbol{D}' \right)_{\operatorname{diag}(a_1,\ldots,a_r)\boldsymbol{u}_2} \;\middle|\; \boldsymbol{u}_1 \in [0,1)^n, \boldsymbol{u}_2 \in [0,1)^r, \operatorname{diag}(a_1,\ldots,a_r)\boldsymbol{u}_2 \in \mathbb{Z}^r \right\} \\
&= \left\{ \mathcal{O}_{\mathcal{X}'}\left(\lfloor a^t B_1 \boldsymbol{u}_1 \rfloor \boldsymbol{D}' \right)_{\boldsymbol{l}} \;\middle|\; \boldsymbol{u}_1 \in [0,1)^n, \boldsymbol{l} \in \mathbb{Z}^r \cap \prod_{i=1}^r [0, a_i) \right\}, \\
\mathfrak{D}_{\mathcal{X}'^{\mathrm{rig}}} &= \left\{ \mathcal{O}_{\mathcal{X}'^{\mathrm{rig}}}\left(\lfloor a^t B_1 \boldsymbol{u}_1 \rfloor \boldsymbol{D}' \right) \;\middle|\; \boldsymbol{u}_1 \in [0,1)^n \right\}.
\end{aligned}$$

Hence we have $\mathfrak{D}_{\mathcal{X}'} = \bigcup_{\boldsymbol{l}} (\Psi'^*\mathfrak{D}_{\mathcal{X}'^{\mathrm{rig}}}) \otimes \mathcal{O}_{\mathcal{X}', \boldsymbol{l}}$, where $\boldsymbol{l}$ runs over the set $\mathbb{Z}^r \cap \prod_{i=1}^r [0, a_i)$, and we define $\Psi'^*\mathfrak{D}_{\mathcal{X}'^{\mathrm{rig}}}$ in an obvious way.

Lemma 4.6. *Let $\mathcal{X}$ be a toric DM stack and $\mathcal{X}'^{rig}$ the rigidification of the root stack $\mathcal{X}' = \sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathcal{X}}$ as above. If $\mathfrak{D}_{\mathcal{X}'^{rig}}$ generates $D^b(\mathcal{X}'^{rig})$, then $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$.*

Proof. Note that by Lemmas 3.5 and 4.4, it is enough to show that $\mathfrak{D}_{\mathcal{X}'}$ generates $D^b(\mathcal{X}')$. The assumption and Theorem 3.1 implies that the set $\mathfrak{D}_{\mathcal{X}'} = \bigcup_{\boldsymbol{l}} (\Psi'^*\mathfrak{D}_{\mathcal{X}'^{\mathrm{rig}}}) \otimes \mathcal{O}_{\mathcal{X}', \boldsymbol{l}}$ generates $D^b(\mathcal{X}')$. Hence we obtain the assertion. □

5 Root stacks of the projective plane

5.1 Root stacks of weighted projective spaces

First let us recall the definition of weighted projective space $\mathbb{P}(\boldsymbol{a})$ for $\boldsymbol{a} = {}^t(a_1, \ldots, a_{n+1}) \in \mathbb{Z}_{>0}^{n+1}$. Take a finitely generated abelian group $N := \mathbb{Z}^{n+1}/\mathbb{Z}\boldsymbol{a}$ and put β to be the quotient map $\mathbb{Z}^{n+1} \to N$. Let us take the canonical basis $\boldsymbol{e}_i$'s of $\mathbb{Z}^{n+1}$ and consider the fan Δ in $N_{\mathbb{R}}$, consisting of cones $\sum_{i \neq j} \mathbb{R}_{\geq 0} \beta_{\mathbb{R}}(\boldsymbol{e}_i)$ for $j = 1, \ldots, n+1$ and their faces. We often denote by $\mathbb{P}(\boldsymbol{a})$ the toric DM stack associated to the stacky fan $\Sigma = (\Delta, \beta)$. This is called a *weighted projective space*.

For $\boldsymbol{b} = {}^t(b_1, \ldots, b_{n+1}) \in \mathbb{Z}_{>0}$, we consider a root stack $\mathcal{X} := \sqrt[\boldsymbol{b}]{\boldsymbol{D}/\mathbb{P}(\boldsymbol{a})}$. Take a projective resolution of N

$$0 \to \mathbb{Z} \xrightarrow{\boldsymbol{a}} \mathbb{Z}^{n+1} \to N = \mathbb{Z}^{n+1}/\mathbb{Z}\boldsymbol{a} \to 0.$$

Then in terms in §2.2, $\mathcal{X}$ is associated with matrices

$$A = \boldsymbol{a} \in M(n+1,1), \quad B = \operatorname{diag}\left(b_1, \dots, b_{n+1}\right) \in M(n+1, n+1).$$

Assume furthermore that $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c} = {}^t(c, \dots, c) \in \mathbb{Z}^{n+1}$ for an integer $c \in \mathbb{Z}$. According to (8), we have

$$\operatorname{Pic} \sqrt[\boldsymbol{c}]{\boldsymbol{D}/\mathbb{P}(\boldsymbol{c})} \cong \frac{\mathbb{Z}\boldsymbol{f}_1^* \oplus \cdots \oplus \mathbb{Z}\boldsymbol{f}_{n+1}^* \oplus \mathbb{Z}\boldsymbol{g}^*}{\langle c\boldsymbol{f}_1^* + c\boldsymbol{g}^*, \dots, c\boldsymbol{f}_{n+1}^* + c\boldsymbol{g}^* \rangle} \quad \mathcal{O}_{\mathcal{X}}(\textstyle\sum l_i \mathcal{D}_i)_{k\bar{\boldsymbol{g}^*}} \mapsto \textstyle\sum l_i \bar{\boldsymbol{f}}_i^* + k\bar{\boldsymbol{g}}^*,$$

and hence we have an isomorphism

$$h\colon \operatorname{Pic} \sqrt[\boldsymbol{c}]{\boldsymbol{D}/\mathbb{P}(\boldsymbol{c})} \to \mathbb{Z} \oplus \mathbb{Z}_c^{\oplus n+1} \quad \sum l_i \bar{\boldsymbol{f}}_i^* + k\bar{\boldsymbol{g}}^* \mapsto \begin{pmatrix} \sum l_i - k \\ \bar{l}_1 \\ \vdots \\ \bar{l}_{n+1} \end{pmatrix}. \tag{20}$$

For any $\mathcal{L} \in \operatorname{Pic} \sqrt[\boldsymbol{c}]{\boldsymbol{D}/\mathbb{P}(\boldsymbol{c})}$, we call the first component $\sum l_i - k$ of $h(\mathcal{L})$ the *degree* of $\mathcal{L}$ and denote it by $\deg \mathcal{L}$.

5.2 Root stacks of the projective plane

Consider the root stack $\mathcal{X} = \sqrt[\boldsymbol{b}]{\boldsymbol{D}/\mathbb{P}^2} = \mathcal{X}(\mathbb{P}^2, b_1D_1 + b_2D_2 + b_3D_3)$, where $\boldsymbol{b} = {}^t(b_1.b_2, b_3)$ and $D_i = [z_i = 0]$ in $\mathbb{P}^2$. The map

$$\beta\colon \mathbb{Z}^3 \to N = \mathbb{Z}^2$$

is given by the matrix

$$B = \begin{pmatrix} b_1 & 0 & -b_3 \\ 0 & b_2 & -b_3 \end{pmatrix}.$$

Moreover we put an additional assumption that $b_1 = b_2 = b_3 = c$. In this case by (19), we have

$$\mathfrak{D}_{\mathcal{X}} = \left\{ \mathcal{O}_{\mathcal{X}}\left(i(\mathcal{D}_1 - \mathcal{D}_3) + j(\mathcal{D}_2 - \mathcal{D}_3) + k\mathcal{D}_3 \right) \mid i, j \in [0, c), k = 0, -1, -2 \right\}.$$

By [BHu, Proposition 5.1], this set forms a full strong exceptional collection on $\mathcal{X}$. In particular we know that $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$.

6 Weighted blow ups and Frobenius morphisms

6.1 Weighted blow ups of toric DM orbifolds

Let us consider a toric DM orbifold $\mathcal{X} = \mathcal{X}_\Delta$ associated with a stacky fan $\Sigma = (\Delta, \beta)$. Define primitive vectors $\boldsymbol{v}_i$ in N for $i = 1, \dots, s$ so that $\rho_i = \mathbb{R}_{\geq 0}\boldsymbol{v}_i$, namely, there exists $b_i \in \mathbb{Z}_{>0}$ such

that $b_i \boldsymbol{v}_i = \beta(\boldsymbol{f}_i)$. Then as in §2.1, $\mathcal{X}$ is also described as $\mathcal{X}\big(X, \sum_{i=1}^{s} b_i D_i\big)$. Take a cone $\sigma \in \Delta$ spanned by $\boldsymbol{v}_1, \dots, \boldsymbol{v}_l$, and another primitive vector $\boldsymbol{v}_{s+1}$ which is in the relative interior of σ; there exist positive integers $h_i, m \in \mathbb{Z}_{>0}$ with coprime h_i's satisfying

$$m\boldsymbol{v}_{s+1} = \sum_{i=1}^{l} h_i \boldsymbol{v}_i.$$

Then we have a new simplicial fan Δ' which is the subdivision of Δ obtained from the the star shaped decomposition of the fan Δ, by adding the ray $\mathbb{R}_{\geq 0}\boldsymbol{v}_{s+1}$. Then we obtain a *weighted blow up* of a toric variety $X = X_\Delta$:

$$\Psi\colon X' = X_{\Delta'} \to X.$$

Suppose that positive integers b_{s+1} and c_i satisfy

$$b_{s+1}\boldsymbol{v}_{s+1} = \sum_{i=1}^{l} c_i b_i \boldsymbol{v}_i. \tag{21}$$

Then define maps γ_1, γ_2 as

$$\gamma_1\colon \bigoplus_{i=1}^{s+1} \mathbb{Z}\boldsymbol{f}'_i \to \bigoplus_{i=1}^{s} \mathbb{Z}\boldsymbol{f}_i \quad \boldsymbol{f}'_i \mapsto \begin{cases} \boldsymbol{f}_i & \text{if } i \leq s \\ \sum_{i=1}^{l} c_i \boldsymbol{f}_i & \text{if } i = s+1 \end{cases}$$

and $\gamma_2 = \mathrm{id}_N$, and take a toric DM orbifold $\mathcal{X}' = \mathcal{X}_{\Sigma'}$ associated with the stacky fan $\Sigma' = (\Delta', \beta \circ \gamma_1)$. Note that $b_{s+1}\boldsymbol{v}_{s+1} = \beta \circ \gamma_1(\boldsymbol{f}'_{s+1})$. Then we obtain a morphism

$$\psi\colon \mathcal{X}' \to \mathcal{X},$$

called *weighted blow-up* of a toric DM orbifold $\mathcal{X}$.

Lemma 6.1. *We have $\mathbb{R}\psi_* \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}}$.*

Proof. The assertion is shown in the proof of [BH1, Theorem 9.1]. ∎

Take a minimum integer $h > 0$ such that each $h\frac{h_i}{b_i}$ is an integer. We define $b_{s+1} := hm$ and $c_i := h\frac{h_i}{b_i}$. Then the equation (21) is achieved, and hence we obtain a morphism $\psi\colon \mathcal{X}' \to \mathcal{X}$ as above. We call it the *weighted blow-up of $\mathcal{X}$ associated with Ψ.*

Remark 6.2. *In the two dimensional case, recall that a torus equivariant resolution*

$$\Theta\colon Y \to X$$

of singularities of X is a t-times composition of weighted blow-ups (cf. [Fu, §2.6]). Hence for such a resolution Θ, we can construct an associated morphism of toric DM orbifolds:

$$\theta\colon \mathcal{Y} = \mathcal{X}(Y, \sum^{s+t} b_i D_i) \to \mathcal{X} = \mathcal{X}(X, \sum^{s} b_i D_i).$$

6.2 Weighted blow ups of two dimensional toric DM stacks and Frobenius morphisms

We consider a two dimensional toric DM orbifold $\mathcal{X} = \mathcal{X}_\Sigma = \mathcal{X}(X, \sum_{i=1}^s b_i D_i)$ associated with a stacky fan $\Sigma = (\Delta, \beta)$ in a free abelian group $N = \mathbb{Z}\boldsymbol{e}_1 \oplus \mathbb{Z}\boldsymbol{e}_2$.

Take a non-singular cone $\sigma \in \Delta$ generated by vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and consider the new simplicial fan Δ' which is the subdivision of Δ obtained from the the star shaped decomposition of σ, by adding the ray $\mathbb{R}_{\geq 0}\boldsymbol{v}_{s+1}$ in which we put $\boldsymbol{v}_{s+1} := \boldsymbol{v}_1 + \boldsymbol{v}_2$. Consider the (weighted) blow-up $\Psi\colon X' \to X$ associated with the subdivision. We may assume that $\boldsymbol{v}_1 = \boldsymbol{e}_1$ and $\boldsymbol{v}_2 = \boldsymbol{e}_2$, and assume furthermore that $b_1 = b_2$, which we denote by c. Let us define the blow-up $\psi\colon \mathcal{X}' \to \mathcal{X}$ associated with Ψ. Then we have

$$\psi^*\mathcal{D}_i = \mathcal{D}'_i + \mathcal{D}'_{s+1} \text{ for } i = 1, 2, \text{ and } \psi^*\mathcal{D}_i = \mathcal{D}'_i \text{ for } i \neq 1, 2. \tag{22}$$

Moreover we have $b_{s+1} = c$. By $\boldsymbol{v}_{s+1} = \boldsymbol{e}_1 + \boldsymbol{e}_2$, we have an isomorphism

$$\mathcal{D}'_{s+1} \cong \sqrt[c]{\boldsymbol{D}/\mathbb{P}(c,c)}$$

for $\boldsymbol{c} = {}^t(c, c)$ by Lemma 2.3. Let us define $\mathcal{Q} = [z_1 = z_2 = 0]$. Then we have the following diagram:

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "\psi"] & \mathcal{X} \\ \mathcal{D}_{s+1} \arrow[u, hook, "j'"] \arrow[r, "p"] & \mathcal{Q}. \arrow[u, hook, "j"'] \end{tikzcd}$$

Lemma 6.3. *There exists a semi-orthogonal decomposition*

$$D^b(\mathcal{X}') = \langle j'_*\{\mathcal{L} \in \operatorname{Pic}\mathcal{D}_{s+1} \mid \deg\mathcal{L} = -1\}, \mathbb{L}\psi^* D^b(\mathcal{X})\rangle.$$

Proof. Let us denote the category in the r.h.s. by $\mathcal{T}$. We show below that any objects in ${}^\perp\mathcal{T}$ are isomorphic to 0. This completes the proof, since $\mathcal{T}$ is an admissible subcategory of $D^b(\mathcal{X}')$.

By Lemma 2.3, we have an isomorphism

$$\mathcal{Q} \cong \left[Q/\left(\mathbb{Z}_c \times \mathbb{Z}_c\right)\right],$$

where Q is a point. By a similar argument in [Or, Theorem 4.3], we see that for an object $A \in {}^\perp\mathcal{T}$, there exists an object B of $D^b(\mathcal{Q})$ satisfying

$$\mathbb{L}j'^*A = p^*B, \tag{23}$$

and that $B \cong 0$ implies $A \cong 0$. We have

$$\mathcal{O}_{\mathcal{Q},\boldsymbol{i}} = j^*\mathcal{O}_{\mathcal{X}}(i_1\mathcal{D}_1 + i_2\mathcal{D}_2)$$

for $\boldsymbol{i} = {}^t(i_1, i_2) \in \mathbb{Z}^2$. Since every object of $D^b(\mathcal{Q})$ is a direct sum of $\mathcal{O}_{\mathcal{Q},\boldsymbol{i}}[l]$ for some $l \in \mathbb{Z}$, it is enough to show $\mathrm{Hom}_{\mathcal{Q}}(B, \mathcal{O}_{\mathcal{Q},\boldsymbol{i}}[l]) = 0$ for any $\boldsymbol{i}$ and l. Take an invertible sheaf

$$\mathcal{F} := \mathcal{O}_{\mathcal{D}'_{s+1}}(i_1\mathcal{D}'_1 + i_2\mathcal{D}'_2)_{i_1+i_2}$$

on $\mathcal{D}'_{s+1}$, and then we have

$$\mathbb{R}p_*\mathcal{F} \cong \mathcal{O}_{\mathcal{Q},\boldsymbol{i}}, \quad j'_*\mathcal{F} \cong \mathbb{L}\psi^* j_*\mathcal{O}_{\mathcal{Q},\boldsymbol{i}}. \tag{24}$$

The first isomorphism can be checked by similar computations in [BHu, Proposition 4.1], and the second is directly proved by the use of (11) and the Koszul resolution of $j_*\mathcal{O}_{\mathcal{Q},\boldsymbol{i}}$:

$$\begin{aligned}0 \to \mathcal{O}_{\mathcal{X}}\left((i_1-1)\mathcal{D}_1 + (i_2-1)\mathcal{D}_2\right) \to \mathcal{O}_{\mathcal{X}}\left((i_1-1)\mathcal{D}_1 + i_2\mathcal{D}_2\right) \oplus \mathcal{O}_{\mathcal{X}}\left(i_1\mathcal{D}_1 + (i_2-1)\mathcal{D}_2\right)\\ \to \mathcal{O}_{\mathcal{X}}\left(i_1\mathcal{D}_1 + i_2\mathcal{D}_2\right) \to j_*\mathcal{O}_{\mathcal{Q},\boldsymbol{i}} \to 0.\end{aligned}$$

By $p^* \dashv \mathbb{R}p_*, \mathbb{L}j'^* \dashv j'_*$, (23) and (24), we get equalities

$$\mathrm{Hom}_{\mathcal{Q}}(B, \mathcal{O}_{\mathcal{Q},\boldsymbol{i}}[l]) = \mathrm{Hom}_{\mathcal{D}_{s+1}}(\mathbb{L}j'^*A, \mathcal{F}[l]) = \mathrm{Hom}_{\mathcal{X}'}(A, \mathbb{L}\psi^*j_*\mathcal{O}_{\mathcal{Q},\boldsymbol{i}}[l]) = 0,$$

since we have $\mathbb{L}\psi^*j_*\mathcal{O}_{\mathcal{Q},\boldsymbol{i}} \in \mathcal{T}$. This implies $B \cong 0$, which completes the proof. □

Lemma 6.4. *If $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$, then $\mathfrak{D}_{\mathcal{X}'}$ generates $D^b(\mathcal{X}')$.*

Proof. For $\boldsymbol{u} \in \mathbb{R}^2$, we define

$$\mathcal{D}_{\boldsymbol{u}} := \sum_{i=1}^{s} \lfloor(\boldsymbol{u}, b_i\boldsymbol{v}_i)\rfloor \mathcal{D}_i, \quad \mathcal{D}'_{\boldsymbol{u}} := \sum_{i=1}^{s+1} \lfloor(\boldsymbol{u}, b_i\boldsymbol{v}_i)\rfloor \mathcal{D}'_i.$$

Recall that $\mathfrak{D}_{\mathcal{X}} = \{\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{\boldsymbol{u}}) \mid \boldsymbol{u} \in [0,1)^2\}$ and $\mathfrak{D}_{\mathcal{X}'} = \{\mathcal{O}_{\mathcal{X}'}(\mathcal{D}'_{\boldsymbol{u}}) \mid \boldsymbol{u} \in [0,1)^2\}$ by (19).

Let $\mathcal{T}'$ denote $j'_*\{\mathcal{L} \in \mathrm{Pic}\,\mathcal{D}'_{s+1} \mid \deg\mathcal{L} = -1\}$, and first we show that $\mathcal{T}' \subset \langle\mathfrak{D}_{\mathcal{X}'}\rangle$. Note that

$$h(\mathcal{O}_{\mathcal{D}'_{s+1}}\left(l_1\mathcal{D}'_1 + l_2\mathcal{D}'_2 + k\mathcal{D}'_{s+1}\right)) = \begin{pmatrix} l_1 + l_2 - k \\ \bar{l}_1 \\ \bar{l}_2 \end{pmatrix} \in \mathbb{Z} \oplus \mathbb{Z}_c \oplus \mathbb{Z}_c$$

for the isomorphism h in (20), hence

$$\mathcal{O}_{\mathcal{D}'_{s+1}}\left(\mathcal{D}'_{\boldsymbol{u}}\right) \cong \mathcal{O}_{\mathcal{D}'_{s+1}}\left(\lfloor cu_1\rfloor\mathcal{D}'_1 + \lfloor cu_2\rfloor\mathcal{D}'_2 + \lfloor cu_1 + cu_2\rfloor\mathcal{D}'_{s+1}\right) \in \mathcal{T}'$$

if

$$\lfloor cu_1\rfloor + \lfloor cu_2\rfloor - \lfloor cu_1 + cu_2\rfloor = -1. \tag{25}$$

Fix $l_1, l_2 \in \mathbb{Z}$, and take a generic element $\boldsymbol{u}$ in the set

$$\left\{ \boldsymbol{u} = {}^t(u_1, u_2) \in \mathbb{R}^2 \;\middle|\; l_1 < cu_1 < l_1 + 1, l_2 < cu_2 < l_2 + 1, cu_1 + cu_2 = l_1 + l_2 + 1 \right\}.$$

Then for a sufficiently small real number $\varepsilon > 0$, we have

$$\mathcal{O}_{\mathcal{X}'}\left(\mathcal{D}'_{\boldsymbol{u} - \epsilon^t(1,1)}\right) = \mathcal{O}_{\mathcal{X}'}\left(\mathcal{D}'_{\boldsymbol{u}} - \mathcal{D}'_{s+1}\right).$$

It follows from the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}'}\left(\mathcal{D}'_{\boldsymbol{u}} - \mathcal{D}'_{s+1}\right) \to \mathcal{O}_{\mathcal{X}'}\left(\mathcal{D}'_{\boldsymbol{u}}\right) \to \mathcal{O}_{\mathcal{D}'_{s+1}}\left(\mathcal{D}'_{\boldsymbol{u}}\right) \to 0 \tag{26}$$

that $\mathcal{O}_{\mathcal{D}'_{s+1}}\left(\mathcal{D}'_{\boldsymbol{u}}\right) \in \langle \mathfrak{D}_{\mathcal{X}'} \rangle$. Since $\boldsymbol{u}$ satisfies (25), we obtain $\mathcal{T}' \subset \langle \mathfrak{D}_{\mathcal{X}'} \rangle$.

Next we show $\mathbb{L}\psi^*\left(D^b(\mathcal{X})\right) \subset \langle \mathfrak{D}_{\mathcal{X}'} \rangle$. By the assumption $\langle \mathfrak{D}_{\mathcal{X}} \rangle = D^b(\mathcal{X})$, it suffices to show that $\psi^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{\boldsymbol{u}}) \in \langle \mathfrak{D}_{\mathcal{X}'} \rangle$ for any $\boldsymbol{u} \in \mathbb{R}^2$. By (22), we have

$$\psi^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{\boldsymbol{u}}) = \begin{cases} \mathcal{O}_{\mathcal{X}'}(\mathcal{D}'_{\boldsymbol{u}}) & \text{if } \lfloor (\boldsymbol{u}, c\boldsymbol{v}_{s+1}) \rfloor = \lfloor cu_1 \rfloor + \lfloor cu_2 \rfloor, \\ \mathcal{O}_{\mathcal{X}'}(\mathcal{D}'_{\boldsymbol{u}} - \mathcal{D}'_{s+1}) & \text{if } \lfloor (\boldsymbol{u}, c\boldsymbol{v}_{s+1}) \rfloor = \lfloor cu_1 \rfloor + \lfloor cu_2 \rfloor + 1. \end{cases}$$

Hence it is enough to consider the second case. Then we see by (25) that $\mathcal{O}_{\mathcal{D}'_{s+1}}\left(\mathcal{D}'_{\boldsymbol{u}}\right) \in \mathcal{T}' \subset \langle \mathfrak{D}_{\mathcal{X}'} \rangle$ and hence $\psi^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{\boldsymbol{u}}) \in \langle \mathfrak{D}_{\mathcal{X}'} \rangle$ by (26).

Now Lemma 6.3 completes the proof. □

7 Proof of Main theorem

The purpose of this note is to show the following.

Theorem 7.1. *For every two dimensional toric DM stack $\mathcal{X}$, the set $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$.*

Proof. By Lemma 4.6, we can reduce the statement to the case $\mathcal{X}$ is an orbifold. Then there exist integers $b_i \in \mathbb{Z}_{>0}$ such that $\mathcal{X} = \mathcal{X}\left(X, \sum_{i=1}^{s} b_i D_i^X\right)$ as in §2.1.

Consider a torus equivariant resolution of X and the associated birational morphism between toric DM orbifolds as in Remark 6.2. Then by Lemmas 4.4 and 6.1 we can reduce to the case X is smooth.

We have a root construction morphism

$$\mathcal{X}' := \mathcal{X}\left(X, c\sum_{i=1}^{s} D_i^X\right) \to \mathcal{X} = \mathcal{X}\left(X, \sum_{i=1}^{s} b_i D_i^X\right)$$

for $c = b_1 b_2 \cdots b_s$. By lemmas 4.4 and 3.5, it suffices to check the statement for $\mathcal{X}'$.

The strong factorization theorem ([Od, Theorem 1.28 (2)]) implies that there exists a smooth complete toric surface Z and morphisms Φ, Ψ;

$$X \xleftarrow{\Phi} Z \xrightarrow{\Psi} \mathbb{P}^2,$$

where Φ and Ψ are compositions of torus equivariant blow-ups. Associated with Φ and Ψ, we obtain a compositions of blow-ups on toric DM orbifolds

$$\mathcal{X}' \xleftarrow{\phi} \mathcal{X}\left(Z, c\sum_{i=1}^{t} D_i^Z\right) \xrightarrow{\psi} \mathcal{X}\left(\mathbb{P}^2, cD_1^{\mathbb{P}^2} + cD_2^{\mathbb{P}^2} + cD_3^{\mathbb{P}^2}\right).$$

By Lemmas 4.4, 6.1 and 6.4, it suffices to show the statement in the case where $\mathcal{X}$ is the root stack $\mathcal{X}\left(\mathbb{P}^2, cD_1^{\mathbb{P}^2} + cD_2^{\mathbb{P}^2} + cD_3^{\mathbb{P}^2}\right)$ of the projective plane $\mathbb{P}^2$, which is already shown in §5.2. $\square$

Remark 7.2. *Suppose that $\mathcal{X}$ is a toric DM stack and consider the following Cartesian diagram:*

$$\begin{tikzcd}
\mathcal{X} \times \mathbb{P}^n \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathbb{P}^n \arrow[d, "q_2"] \\
\mathcal{X} \arrow[r, "q_1"'] & \operatorname{Spec} \mathbb{C}
\end{tikzcd}$$

Then we have

$$\mathbb{R}p_{2*}\mathcal{O}_{\mathcal{X}\times\mathbb{P}^n} = \mathbb{R}p_{2*}p_1^*\mathcal{O}_{\mathbb{P}^n} = q_1^*\mathbb{R}q_{2*}\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_{\mathcal{X}}$$

by the flat base change theorem. Hence by Lemma 4.4, if $\mathfrak{D}_{\mathcal{X}\times\mathbb{P}^n}$ generates $D^b(\mathcal{X}\times\mathbb{P}^n)$, then $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$. In particular, $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$ for 0 or 1-dimensional toric DM stacks $\mathcal{X}$ by Theorem 7.1.

8 Full strong exceptional collections

In this section, we choose a full strong exceptional collection from the set $\mathfrak{D}_{\mathcal{X}}$, in several examples of one or two dimensional toric stacks $\mathcal{X}$.

8.1 Examples

First let us show Lemma 8.1. Let $\mathcal{X}$ be a toric DM orbifold and assume that a coarse moduli space X of $\mathcal{X}$ is smooth. Recall that we have a root construction morphism $\pi\colon \mathcal{X} \to X$ as in §3.3. Then for any divisor $\mathcal{D}$ on $\mathcal{X}$, there exists a divisor D on X such that $\pi^*\mathcal{O}_X(D) = \mathcal{O}_{\mathcal{X}}(\mathcal{D})^{\otimes m}$ for some $m > 0$. Then we know that $\mathcal{D}$ is nef on $\mathcal{X}$ if and only if so is D on X. Under this notation, we have the following:

Lemma 8.1. *If $\mathcal{D}$ is nef, then we have $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) = 0$ for $i > 0$.*

Proof. Since $\mathcal{O}_{\mathcal{X}}$ is a direct summand of $F_{m*}\mathcal{O}_{\mathcal{X}}$, we have

$$\begin{aligned} H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \subset & H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \otimes F_{m*}\mathcal{O}_{\mathcal{X}}) \cong H^i(\mathcal{X}, F_m^*\mathcal{O}_{\mathcal{X}}(\mathcal{D})) \\ \cong & H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})^{\otimes m}) \cong H^i(\mathcal{X}, \pi^*\mathcal{O}_X(D)) = 0, \end{aligned}$$

which completes the proof. Note that the last equality is a consequence of the nef vanishing on toric varieties. □

Put

$$\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} := \{\mathcal{O}_{\mathcal{X}}(\mathcal{D}) \in \mathfrak{D}_{\mathcal{X}} \mid -\mathcal{D} \text{ is nef}\}.$$

If we have

$$\mathfrak{D}_{\mathcal{X}} \subset \langle \mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} \rangle,$$

then the set $\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}}$ forms a full strong exceptional collection by [Ue, Lemma 3.8(i)], Lemma 8.1 and Theorem 7.1.

Below we give calculation only in some typical cases and omit it in the rest. We leave it to readers.

(1) For $\mathcal{X} = \mathbb{P}(a_1, a_2, a_3)$ with $(a_1, a_2, a_3) = 1$ or $\mathcal{X} = \mathcal{X}(\mathbb{P}^1 \times \mathbb{P}^1, \sum b_i D_i)$ with arbitrary $b_1, b_2, b_3, b_4 \in \mathbb{Z}_{>0}$, we have $\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} = \mathfrak{D}_{\mathcal{X}}$. Hence $\mathfrak{D}_{\mathcal{X}}$ forms a full strong exceptional collection.

(2) Blow-up at a point on $\mathbb{P}^2$, and then we obtain the Hirzebruch surface $\mathbb{F}_1$. For the toric DM orbifold $\mathcal{X} = \mathcal{X}(\mathbb{F}_1, D_1 + D_2 + 2D_3 + D_4)$, we have $\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} \neq \mathfrak{D}_{\mathcal{X}}$, where D_1, D_3 are fiber of the ruling and D_2, D_4 are negative and positive sections respectively. In fact we take vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \text{ and } \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in N,$$

corresponding to divisors $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{D}_4$ on $\mathcal{X}$. Then we obtain

$$\mathfrak{D}_{\mathcal{X}} = \{\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_3), \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3), \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_4), \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_4), \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_3 - \mathcal{D}_4), \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4)\}$$

and $\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} = \mathfrak{D}_{\mathcal{X}} \setminus \{\mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_4)\}$. In this case we can see that $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_4) \in \langle \mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} \rangle$, and hence $\mathfrak{D}_{\mathcal{X}} \subset \langle \mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} \rangle$.

(3) Finally we take the toric DM orbifold $\mathcal{X}$ defined by vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{v}_5 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

in N corresponding to divisors $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ and $\mathcal{D}_5$ on $\mathcal{X}$. Then we obtain

$$\begin{aligned} \mathfrak{D}_{\mathcal{X}} = & \{\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_3 - \mathcal{D}_4), \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4), \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_5), \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_5), \\ & \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4 - \mathcal{D}_5), \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_3 - \mathcal{D}_4 - \mathcal{D}_5), \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_4 - \mathcal{D}_5), \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_4 - \mathcal{D}_5)\} \end{aligned}$$

and $\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} = \mathfrak{D}_{\mathcal{X}} \setminus \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$, where we define

$$\mathcal{L}_1 := \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_5), \mathcal{L}_2 := \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_3 - \mathcal{D}_4) \text{ and } \mathcal{L}_3 := \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_4 - \mathcal{D}_5).$$

In this case, since $\#\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} = 6 < \operatorname{rk} K(\mathcal{X}) = 7$, the set $\mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}}$ does not form a full exceptional collection (see Remark 4.3), and thus we know $\mathfrak{D}_{\mathcal{X}} \not\subset \langle \mathfrak{D}_{\mathcal{X}}^{\mathrm{nef}} \rangle$. However we can see the subset

$$\mathcal{S} := \mathfrak{D}_{\mathcal{X}} \setminus \{\mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4), \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4 - \mathcal{D}_5)\}$$

of $\mathfrak{D}_{\mathcal{X}}$ is a full strong exceptional collection as follows.

By the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3 - \mathcal{D}_5) \to \mathcal{O}_{\mathcal{D}_3}(\mathcal{D}_3) \to 0,$$

we have $\mathcal{O}_{\mathcal{D}_3}(-\mathcal{D}_3 - \mathcal{D}_4) = \mathcal{O}_{\mathcal{D}_3}(\mathcal{D}_3) \in \langle \mathcal{S} \rangle$. Moreover by the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4 - i\mathcal{D}_5) \to \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_3 - \mathcal{D}_4 - i\mathcal{D}_5) \to \mathcal{O}_{\mathcal{D}_3}(-\mathcal{D}_3 - \mathcal{D}_4) \to 0 \qquad (27)$$

for $i = 0, 1$, we have $\mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4 - i\mathcal{D}_5) \in \langle \mathcal{S} \rangle$. Hence we obtain $\mathfrak{D}_{\mathcal{X}} \subset \langle \mathcal{S} \rangle$ and hence $\mathcal{S}$ generates $D^b(\mathcal{X})$. According to [BHu, Proposition 4.1], we show below that for any $\mathcal{L}, \mathcal{L}' \in \mathcal{S}$, $\operatorname{Ext}^i_{\mathcal{X}}(\mathcal{L}, \mathcal{L}') = 0$ for $i \neq 0$ to conclude that $\mathcal{S}$ is a full strong exceptional collection.

For every $\boldsymbol{r} = (r_i) \in \mathbb{Z}^5$, we denote by $\mathrm{Supp}(\boldsymbol{r})$ the simplicial complex on five vertices $\{1, \ldots, 5\}$ which consists of all subsets $J \subset \{1, \ldots, 5\}$ such that $r_i \geq 0$ for all $i \in J$ and there exists a cone in the fan Δ determining $\mathcal{X}$ that contains all $\boldsymbol{v}_i, i \in J$. By [BHu, Proposition 4.1] [2] we see that for any line bundle $\mathcal{L}$ on $\mathcal{X}$, we have

$$\dim H^1(\mathcal{X}, \mathcal{L}) = \sum_{\boldsymbol{r}} (\#\{\text{ connected components of } \mathrm{Supp}(\boldsymbol{r})\} - 1), \qquad (28)$$

where the summation is taken over the set of all $\boldsymbol{r} = (r_i) \in \mathbb{Z}^5$ such that $\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^5 r_i D_i) \cong \mathcal{L}$.

For example, we know from (28) that

$$\operatorname{Ext}^1_{\mathcal{X}}(\mathcal{L}_1, \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4)) = H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-3\mathcal{D}_3 - \mathcal{D}_4 + D_5)) \neq 0,$$

since $-3\mathcal{D}_3 - \mathcal{D}_4 + \mathcal{D}_5 \sim -\mathcal{D}_1 - \mathcal{D}_3 + \mathcal{D}_5$ and $\mathrm{Supp}({}^t(-1, 0, -1, 0, 1))$ has 2 connected components. By [BHu, Proposition 4.1] and easy but tedious computations as above, we can see that $\mathcal{S}$ forms a full strong exceptional collection.

Moreover again by (28) we have

$$\operatorname{Ext}^1_{\mathcal{X}}(\mathcal{L}_i, \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4)) \neq 0 \text{ and } \operatorname{Ext}^1_{\mathcal{X}}(\mathcal{L}_i, \mathcal{O}_{\mathcal{X}}(-2\mathcal{D}_3 - \mathcal{D}_4 - \mathcal{D}_5)) \neq 0$$

for all $i = 1, 2, 3$. Hence we conclude that $\mathcal{S}$ is a unique subset of $\mathfrak{D}_{\mathcal{X}}$ which forms a full exceptional collection.

We remark that any two dimensional toric Fano orbifold has a full strong exceptional collection of line bundles by [BHu, Theorem 7.3].

[2] There is a typo in the statement [BHu, Proposition 4.1]. Precisely, the cohomology $H^p(\mathcal{X}, \mathcal{L})$ is obtained by $(\operatorname{rk} N - p - 1)$-th reduced homology of $\mathrm{Supp}(\boldsymbol{r})$.

8.2 Toric DM orbifolds with $\operatorname{rank}\operatorname{Pic}\mathcal{X} = 1$.

As an application of Main Theorem, we also have the following:

Theorem 8.2. *Let $\mathcal{X}$ be an one or two dimensional toric DM stack. Suppose furthermore that its rigidification $\mathcal{X}^{\mathrm{rig}}$ has the Picard group of rank 1. (Notice that this condition is automatically satisfied when $\mathcal{X}$ is 1-dimensional.) Then the set $\mathfrak{D}_{\mathcal{X}}$ forms a full strong exceptional collection.*

Proof. By Corollary 3.2, it suffices to show the statement for toric DM orbifolds. Take the linear function $\deg\colon \operatorname{Pic}\mathcal{X} \to \mathbb{Z}$ which takes value 1 on the positive generator of $\operatorname{Pic}\mathcal{X}$.

We first show that $\mathfrak{D}_{\mathcal{X}}$ is contained in the set $\mathfrak{L}$ of line bundles $\mathcal{L}$ satisfying $\deg K_{\mathcal{X}} < \deg \mathcal{L} \leq 0$. Take an arbitrary line bundle $\mathcal{L}$ in $\mathfrak{D}_{\mathcal{X}}$. Then by (19), $\mathcal{L}$ is of the form $\mathcal{O}_{\mathcal{X}}\left(\sum_i \lfloor \frac{(\boldsymbol{u}, b_i \boldsymbol{v}_i)}{m} \rfloor \mathcal{D}_i\right)$ for some $m \in \mathbb{Z}_{>0}$ and $\boldsymbol{u} \in \mathbb{Z}^n \cap [0, m-1]^n$. Since

$$(\boldsymbol{u}, b_i \boldsymbol{v}_i) = m \lfloor \frac{(\boldsymbol{u}, b_i \boldsymbol{v}_i)}{m} \rfloor + r_i$$

for some integers r_i with $0 \leq r_i < m$, the divisor $m \sum_i \lfloor \frac{(\boldsymbol{u}, b_i \boldsymbol{v}_i)}{m} \rfloor \mathcal{D}_i$ is linearly equivalent to the divisor $-\sum_i r_i \mathcal{D}_i$. In particular, we have $\mathcal{L}^{\otimes m} \cong \mathcal{O}_{\mathcal{X}}(-\sum_i r_i \mathcal{D}_i)$, and hence we conclude that $\deg K_{\mathcal{X}} < \deg \mathcal{L} \leq 0$.

By [BHu, Proposition 5.1] the set $\mathfrak{L}$ forms a full strong exceptional collection. Since $\mathfrak{D}_{\mathcal{X}}$ generates $D^b(\mathcal{X})$, the set $\mathfrak{D}_{\mathcal{X}}$ coincides with the collection they chose. Therefore the set $\mathfrak{D}_{\mathcal{X}}$ forms a full strong exceptional collection. $\square$

References

[Ac] P. Achinger, *A note on the Frobenius morphism on toric varieties*, arXiv:1012.2021.

[BCS] L. Borisov, L. Chen, G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks.* J. Amer. Math. Soc. 18 (2005), no. 1, 193–215

[Bo] A. I. Bondal, *Derived categories of toric varieties*, Oberwolfach Reports, 3 (1), 284–286, 2006.

[BH1] L. Borisov, R. Horja, *On the K-theory of smooth toric DM stacks, Snowbird lectures on string geometry*, Contemp. Math., 401, 21–42, Amer. Math. Soc., Providence, RI, 2006.

[BH2] L. Borisov, R. Horja, *Mellin-Barnes integrals as Fourier-Mukai transforms*, Adv. Math. 207 (2), 876–927, 2006.

[BHu] L. Borisov, Z. Hua, *On the conjecture of King for smooth toric Deligne-Mumford stacks*, Adv. Math. 221 (1), 277–301, 2009.

[Co] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. 4, 17–50, 1995.

[FMN] B. Fantechi, E. Mann, F. Nironi, *Smooth toric DM stacks*, arXiv:0708.1254.

[Fu] W. Fulton, Introduction to toric varieties. Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993. xii+157 pp.

[HMS] D. Huybrechts, E. Macri, P. Stellari, *Formal deformations and their categorical general fibre.*, Comment. Math. Helv. 86 (1), 41–71, 2011.

[Iw1] I. Iwanari, *Logarithmic geometry, minimal free resolutions and toric algebraic stacks.* Publ. Res. Inst. Math. Sci. 45 (4), 1095–1140, 2009.

[Iw2] I. Iwanari, *The category of toric stacks.* Compos. Math. 145 (3), 718–746, 2009.

[IU] A. Ishii, K. Ueda, *The special McKay correspondence and exceptional collection*, arXiv:1104.2381.

[Ka] Y. Kawamata, *Derived categories of toric varieties.* Michigan Math. J. 54 (2006), no. 3, 517-535.

[Od] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 15. Springer-Verlag, Berlin, 1988. viii+212 pp.

[Or] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves.*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 4, 852–862; translation in Russian Acad. Sci. Izv. Math. 41 (1993), no. 1, 133–141

[Pe] F. Perroni, *A note on toric Deligne-Mumford stacks.* Tohoku Math. J. (2) 60 (3), 441–458, 2008.

[Th] F. Thomsen, *Frobenius direct images of line bundles on toric varieties.* J. Algebra. 226 (2000), 865–874.

[TU] Y. Toda, H. Uehara, *Tilting generators via ample line bundles*, Adv. Math. 233 (2010), 1–29.

[Ue] H. Uehara, *Exceptional collections on toric Fano threefolds and birational geometry*, arXiv:1012.4086.

[Vi] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces.* Invent. Math. 97 (1989), no. 3, 613–670.

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